Math 31B: Week 9 Section

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Discussion Questions

Question 1. We want to figure out whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ converges or diverges.

(a) What is wrong with the following argument?

Since $(\ln(n))^2 \geq 1$, we have that $\dfrac{1}{n(\ln(n))^2} \leq \dfrac{1}{n}$. Hence by the direct comparison test, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ diverges.

(b) What is the integral test?

(c) Use the integral test to determines whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ diverges or converges.

Solution to Question 1.

(a) The inequality is the wrong way for the direct comparison test.

(b) This can be found in the textbook.

(c) We have $f(x) = \dfrac{1}{x(\ln(x))^2}$ is positive and decreasing and the sequence is given by $f(n)$. Hence the integral test applies and we find

$$\int_2^{\infty} f(x)dx = \int_{\ln(2)}^{\infty} \frac{du}{u^2} = \frac{1}{\ln(2)} < \infty.$$

Hence the series converges.

Question 2. We investigate the convergence of the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ where

$$a_n = \begin{cases} \frac{2}{n+1} & \text{when } n \text{ odd} \\ \frac{1}{2^{n/2}} & \text{when } n \text{ even.} \end{cases}$$

i.e,

$$\sum_{n=1}^{\infty} (-1)^n a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{8} + \cdots$$

(a) Prove that this series is not absolutely convergent.

(b) What is wrong with the following argument?

Since this is an alternating series, by the alternating series test the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

(c) Does this series converge or diverge? Hint: Try and sum the negative terms of the series, what are you left with?

Solution to Question 2.

(a) We have that

$$\sum_{n=1}^{\infty} |a_n| \geq \sum_{n=1}^{\infty} a_{2n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Hence the series is not absolutely convergent.

(b) The sequence (a_n) is both positive and has limit zero as $n \to \infty$. However, it is not decreasing so the alternating series test does not apply.

(c) The negative terms form a geometric series with total sum -1 while the positive terms form a harmonic series and so diverges. Hence the series will diverge.

Question 3. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges. You may use that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

Solution to Question 3.
We have $a_n = \dfrac{n!}{n^n}$ and we try the ratio test

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \\ &= \left(\frac{n}{n+1} \right)^n \\ &\to \frac{1}{e} \text{ as } n \to \infty. \end{aligned}$$

Hence by the ratio test, the series converges.

Extra Questions

Question 4. determine convergence or divergence of the following.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$

Solution to Question 4.

(a) We have that $\left|\frac{\sin(n)}{n^2}\right| \leq \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges, by the direct comparison test we see this series converges absolutely and hence converges.

(b) We apply the ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} = \frac{1}{5}\left(1 + \frac{1}{n}\right)^3 \to \frac{1}{5} < 1.$$

Hence converges by ratio test.

(c) The sequence of terms is positive and decreasing, hence the integral test applies. We can use the substitution $u = \sqrt{x}$ to integrate it.

$$\int_1^\infty \frac{1}{x + \sqrt{x}} = \int_1^\infty \frac{2du}{u+1} = \infty.$$

Hence the series diverges by integral test.

(d) For $x > 0$ we have that $\sin(x) \leq x$. Hence $\sin\left(\frac{1}{n^2}\right) \leq \frac{1}{n^2}$ and so by the direct comparison test the series converges.