Math 31B: Week 8 Section

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Information

Discussion Questions

Question 1. The formal definition of limit is one of the hardest things to understand when first encountered.

One interesting way to think of limits is as a game[1]. The game is as follows:
Set up: We have two players A and B as well as a sequence (a_n) and a number L.

1. Player B picks a number $\epsilon > 0$ (preferably small).
2. Player A then picks an integer $M > 0$ (preferably large).
3. Player B then picks an integer N larger than M.

The value $|a_N - L|$ is then checked. If it is larger than ϵ, player B wins. If it is smaller than ϵ, player A wins. Then $\lim_{n\to\infty} a_n = L$ is the same thing as player A can always win, while $\lim_{n\to\infty} a_n \neq L$ means Player B can always win (assuming both players are playing smartly).

(a) Just to get a bit of practice with the game, find a partner and play against them with $a_n = \frac{n+4}{n+1}$ and $L = 1$. Who do you expect to win?

(b) With the same a_n and L as the previous question. Suppose player B picks $\epsilon = 1/5$. What M should player A pick to ensure that they win the game?

(c) Suppose we have the sequence $a_n = (-1)^n$ and $L = 1$. What value for ϵ should player B pick to ensure that he wins?

Solution to Question 1.

(a) Assuming player A doesn't make a mistake, they can always win as $\lim_{n\to\infty} \frac{n+4}{n+1} = 1$.

(b) We want to pick an M such that we always have $|a_n - 1| < \frac{1}{5}$ for any $n > M$. Now, after some rearranging, we find that:

$$\begin{aligned} |a_n - 1| &< \frac{1}{5} \\ \left|\frac{n+4}{n+1} - 1\right| &< \frac{1}{5} \\ \frac{3}{n+1} &< \frac{1}{5} \\ \iff n &> 14. \end{aligned}$$

Hence, if player A picks any $M > 14$, then no matter what N player B picks, it will always be larger than M and hence larger than 14 and so we will have $|a_N - 1| < \frac{1}{5}$. Hence player A wins if they pick, say, $M = 15$ (really any integer > 14 will do). Observe that the same line of reasoning can be done no matter what ϵ player B originally picked. Hence we find that player A has a winning strategy. i.e, $\lim_{n\to\infty} a_n = 1$.

[1] Adapted from https://cs.stanford.edu/people/slingamn/limits.pdf, have a look if you have time

(c) Any number smaller than 2 will do. Suppose player B picks 1. Notice that we always have that

$$|a_n - 1| = \begin{cases} 0 \text{ if } n \text{ even} \\ 2 \text{ if } n \text{ odd.} \end{cases}$$

Hence no matter what number M player A picks, all player B needs to to do is pick N to be some odd number larger than M to ensure that they win.

Question 2. Determine the limit of the following sequences as $n \to \infty$.

(a) $a_n = \sqrt{4 + \frac{1}{n}}$

(b) $a_n = \sqrt{n+3} - \sqrt{n}$

Solution to Question 2.

(a) We have that $\lim_{n\to\infty} 4 + \frac{1}{n} = 4$ by limit laws. Since the square root function $\sqrt{x}$ is continuous for $x > 0$, we have that

$$\lim_{n\to\infty} \sqrt{4 + \frac{1}{n}} = \sqrt{\lim_{n\to\infty} 4 + \frac{1}{n}} = \sqrt{4} = 2.$$

(b) We have that

$$\sqrt{n+3} - \sqrt{n} = \frac{n+3-n}{\sqrt{n+3} + \sqrt{n}} = \frac{3}{\sqrt{n+3} + \sqrt{n}}.$$

Hence

$$\lim_{n\to\infty} \sqrt{n+3} - \sqrt{n} = \lim_{n\to\infty} \frac{3}{\sqrt{n+3} + \sqrt{n}} = 0.$$

Question 3. Use partial fractions to rewrite $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ as a telescoping series and find it's value.

Solution to Question 3.
We have that $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$. Hence we get that

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{k+1}.$$

Hence taking $k \to \infty$ and we get

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Homework Questions

Section 11.1
18, 26, 32, 40, 54, 62, 66, 70, 73, 81, 83
Section 11.2
14, 18. 22, 26, 34, 42, 46, 49, 53, 58, 59

Extra Questions

Question 4. Let a_n be the sequence defined recursively as follows:

$$a_0 = 0, \quad a_{n+1} = \sqrt{2 + a_n}.$$

(a) Show that if $a_n < 2$, then $a_{n+1} < 2$.

(b) Show that if $a_n < 2$, then $a_n \leq a_{n+1}$.

(c) The previous parts imply that the sequence (a_n) is increasing and bounded above since $a_0 < 2$. Hence the sequence has a limit L. Find L by taking the limit of both sides of the recursion equation.

Solution to Question 4.

(a) If $a_n < 2$, then we have that

$$\begin{aligned} a_{n+1} &= \sqrt{2 + a_n} \\ &< \sqrt{2 + 2} \\ &= 2. \end{aligned}$$

(b) If $a_n < 2$, then we have that $\frac{1}{a_n} > \frac{1}{2}$ and $\frac{1}{a_n^2} > \frac{1}{4}$. Hence

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \sqrt{\frac{2}{a_n^2} + \frac{1}{a_n}} \\ &> \sqrt{\frac{2}{4} + \frac{1}{2}} \\ &= 1. \end{aligned}$$

Therefore, $a_{n+1} > a_n$. Note the a_n always positive.

(c) Taking the limit as $n \to \infty$ of both sides of $a_{n+1} = \sqrt{2 + a_n}$ gives us $L = \sqrt{2 + L}$. Solving this gives $L = 2$.

Question 5. Let a and b be digits from 0 to 9. Find a fraction that has repeating decimal expansion given by $0.abababababab\ldots$.

Solution to Question 5.

Written as a series, this number is given by $\sum_{k=0}^{\infty} \left(\frac{a}{10} + \frac{b}{10^2}\right) \frac{1}{10^{2k}}$. This is a geometric series with $c = \left(\frac{a}{10} + \frac{b}{10^2}\right)$ and $r = \frac{1}{10^2}$. Hence we have that this decimal is equal to

$$\left(\frac{a}{10} + \frac{b}{10^2}\right) \frac{1}{1 - 10^2} = \frac{10a + b}{10^2(1 - 10^2)}.$$