Math 31B: Week 6 Section

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Information

Discussion Questions

Question 1.

(a) Find the Taylor polynomial T_3 of $\cos(x)$ centered at $x = \pi/2$.

(b) Prove that the n-th Maclaurin polynomial of $\cos(x)$ is given by

$$T_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}.$$

(c) Use the error bound

$$|T_n(x) - \cos(x)| \leq \frac{K|x-a|^{n+1}}{(n+1)!}$$

to find an n such that

$$|T_n(0.1) - \cos(0.1)| \leq 10^{-7}.$$

Solution to Question 1.

(a) $T_3(x) = -(x - \frac{\pi}{2}) + \frac{1}{3!}(x - \frac{\pi}{2})^3$.

(b) We have that $f^n(0) = \frac{(-1)^n + (-1)^{n+1}}{2}$. i.e, it is zero for odd numbers and alternates between 1 and -1 for even numbers.

(c) K is a number larger than the maximum of the absolute value of the $n+1$-th deriviative over the interval $[0, 0.1]$. Since $|\cos(x)|$ and $|\sin(x)|$ are bounded above by 1 we can just take $K = 1$. Substituting our values into the inequality gives

$$|T_n(0.1) - \cos(0.1)| \leq \frac{1}{10^{n+1}(n+1)!}.$$

We want this less than 10^{-7}. i.e,

$$\frac{1}{10^{n+1}(n+1)!} \leq 10^{-7}.$$

Rearranging gives us that

$$(n+1)! \geq 10^{6-n}$$

which by inspection we can see that we can take $n = 6$ or greater. Ofcourse, we can also see that n=4,5 also work after a bit of calculation.

Question 2.

(a) Show that $\int_1^\infty \frac{dx}{x^3+1}$ converges by comparing it with $\int_1^\infty \frac{dx}{x^3}$.

(b) Show that $\int_e^\infty \frac{dx}{x\ln(x)}$ diverges.

Solution to Question 2.

(a) We have that $0 \leq \frac{1}{x^3+1} \leq \frac{1}{x^3}$ for positive x and $\int \frac{dx}{x^3}$ converges as it's a p-integral. Hence by the comparison test the original integral converges.

(b) We have that $\int_e^a \frac{dx}{x\ln(x)} = \ln(\ln(a))$. Hence as $a \to \infty$ this limit diverges.

Question 3. *Gabriel's Horn.* Let $f(x) = \frac{1}{x}$. We will show that some shapes can have infinite surface area but only finite volume.

(a) Show that the surface of revolution around the x-axis over the interval $[1, \infty)$ is given by

$$2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx.$$

(b) Use the comparison test to show that this integral diverges.

(c) Show that the volume of revolution over this same interval is finite.

Solution to Question 3.

(a) This is a direct application of the surface area formula from last week.

(b) Since $\sqrt{1+\frac{1}{x^4}} \geq 1$, we have that $\frac{1}{x}\sqrt{1+\frac{1}{x^4}} \geq \frac{1}{x}$. Since $\int_1^\infty \frac{1}{x}dx$ diverges, we have that the original integral diverges by the comparison test.

(c) Using the disk method, we find that the volume of revolution is given by

$$\pi \int_1^\infty \frac{1}{x^2}dx.$$

This is a p-integral that converges. i.e, we see that

$$\int_1^a \frac{dx}{x^2} = 1 - \frac{1}{a} \to 1 \text{ as } a \to \infty.$$

Homework Questions

Section 9.4
4, 14, 18, 20, 31, 36, 44, 49, 52
Section 8.7
1, 4, 5, 6, 14, 16, 26, 32, 38, 46, 50, 54, 60, 62, 66, 76

Extra Questions

*** Question 4.** Notice that as a consequence of the error bound[1], we have that the remainder of a taylor expansion around a satisfies $\lim_{x \to a} \frac{R_n(x)}{(x-a)^n} = 0$. This can be used to solve limits in a similar way to L'Hopitals rule.

(a) Use L'hopitals to find the limit $\lim_{x \to 0} \frac{\sin(x) - x}{x^3}$.

(b) We know from the first question that $\sin(x) = x - \frac{x^3}{3!} + R_3(x)$. Use this to find the limit $\lim_{x \to 0} \frac{\sin(x) - x}{x^3}$.

(c) Similarly, find the limit $\lim_{x \to 0} \frac{\ln(1-x) - x}{x^2}$ with Taylor polynomials.

[1] Assuming $f^{(n+1)}(x)$ exists and is continuous.