Math 31B: Week 5 Section

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Information

Discussion Questions

Question 1.

(a) What is the point of numerical integration?

(b) Describe in your own words what the Midpoint (M_N), Trapezoidal (T_N) and Simpson's (S_N) Rules are.

(c) What are the formulas for these rules? (Can you do so without looking at your notes?)

Solution to Question 1.

(a) Not all functions (In some sense, almost all functions) don't have an antiderivative that can be defined in terms of simpler functions that we can easily compute values of (Transcendental functions), so the FTC can't be used in a nice way to get exact values for definite integrals. In these cases we instead have to approximate the integral.

(b) Midpoint rule uses rectangles (or trapezoids tangent to the function), Trapezoid rule uses trapezoids and Simpson's rule uses parabolas.

(c) These can be found in the textbook.

Question 2. Consider the definite integral $\int_2^5 \frac{1}{x} dx$. In this question we will investigate how well the Trapezoidal Rule (T_N) approximates this integral. The error bound is given by the formula

$$\text{error}(T_N) \leq \frac{K_2(b-a)^3}{12N^2}.$$

(a) Do you expect the Trapezoidal Rule to over or underestimate the definite integral? If so, why?

(b) Let $f(x) = \frac{1}{x}$, the constant K_2 is any number such that $|f''(x)| \leq K_2$ for all x in the interval we are integrating over. However we usually take it to be the the absolute value of the maximum of the second derivative, $|\max_{x\in[a,b]} f''(x)|$. Find the maximum of f'' and set K_2 to be the absolute value of this value.

(c) In the formula $b-a$ is the length of the interval we are integrating over. In this case we have $b-1 = 5-2 = 3$. Use this and the previous part to find a value of N for which $\text{error}(T_N) < 10^{-6}$.

Solution to Question 2.

(a) The function is $f(x) = \dfrac{1}{x}$ is convex and so we expect it to overestimate the definite integral.

(b) We have $f''(x) = \dfrac{2}{x^3}$ which is decreasing on the interval $[2,5]$. Hence we take $K_2 = |f''(2)| = \dfrac{1}{4}$.

(c) We want to find N such that $\frac{K_2(b-a)^3}{12N^2} = \frac{2\cdot 3^3}{12N^2} \le 10^{-6}$. Rearranging this gives $N \ge \frac{3}{\sqrt{2}} \times 10^3$ and so any N larger than this will do.

Question 3. Compute the arc length of $y = \ln\left(\frac{e^x+1}{e^x-1}\right)$ over the interval $[1,3]$.

Solution to Question 3.
We have the following:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-2e^x}{e^{2x}-1}\\
\left(\frac{dy}{dx}\right)^2 &= \frac{4e^{2x}}{(e^{2x}-1)^2}\\
1+\left(\frac{dy}{dx}\right)^2 &= \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}\\
\sqrt{1+\left(\frac{dy}{dx}\right)^2} &= \frac{e^{2x}+1}{e^{2x}-1}\\
&= \coth(x) \text{ for } x \ge 0.
\end{aligned}$$

Hence we have that

$$\text{Arclength} = \int_1^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^3 \coth(x)dx = \ln(\sinh(3)) - \ln(\sinh(1)).$$

Question 4. Compute the surface are of revolution about the x-axis for $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$ over the interval $[1,e]$.

Solution to Question 4.
We have the following:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}x - \frac{1}{2x}\\
\left(\frac{dy}{dx}\right)^2 &= \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2}\\
1+\left(\frac{dy}{dx}\right)^2 &= \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2}\\
&= \frac{1}{4}(x+\frac{1}{x})^2\\
\sqrt{1+\left(\frac{dy}{dx}\right)^2} &= \frac{1}{2}(x+\frac{1}{x}) \text{ for } x > 0.
\end{aligned}$$

Hence we have that

$$\begin{aligned}
\text{Surface Area} &= 2\pi\int_1^e y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\\
&= 2\pi\int_1^e \left(\frac{1}{4}x^2 - \frac{1}{2}\ln(x)\right)\left(\frac{1}{2}(x+\frac{1}{x})\right)dx\\
&= \pi\int_1^e \frac{1}{4}x^3 + \frac{1}{4}x - \frac{x\ln(x)}{2} - \frac{\ln(x)}{2x}dx\\
&= \frac{\pi}{16}\left(e^4 - 9\right).
\end{aligned}$$

Homework Questions

Section 8.9
12, 16, 34, 36, 38, 40
Section 9.1
2, 9, 14, 18, 21, 23, 28, 40, 42

Extra Questions

Question 5. Evaluate the following integrals

(a) $\displaystyle\int \frac{dx}{x^2+2x+5}$

(b) $\displaystyle\int \sin^5(x)\cos^2(x)dx$

(c)* $\displaystyle\int \sin^4(x)\cos^2(x)dx$

(d) $\displaystyle\int \sqrt{1+\sqrt{x}}dx$

(e)* $\displaystyle\int \frac{1}{\operatorname{sech}(x)}dx.$

(f)** $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{\cos(x)+\sin(x)}dx$

Hint for (f): Remember the trig identities $\sin(\pi/2-x)=\cos(x)$.

Solution to Question 5.

(a) $\frac{1}{\sqrt{3}}\arctan\left(\frac{x+1}{\sqrt{3}}\right)+C$. Rewrite $\displaystyle\frac{1}{x^2+2x+5}=\frac{1}{3}\cdot\frac{1}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1}$ and use substitution.

(b) By the Pythagorean identity, we have

$$\int \sin^5(x)\cos^2(x)dx=\int \sin(x)(1-\cos^2(x))^2\cos^2(x)dx=\int \left(\cos^2(x)-2\cos^4(x)+\ \cos^6(x)\right)\sin(x)dx.$$

Hence $\displaystyle\int \sin^5(x)\cos^2(x)dx=-\cos^3(x)/3+2\cos^5(x)/5-\cos^7(x)/7+C.$

(c) By the double angle formulas, we have that

$$\begin{aligned}\int\int \sin^4(x)\cos^2(x)dx &= \int \frac{1}{4}\sin^2(2x)\sin^2(x)dx\\ &= \int \frac{1}{4}\left(\frac{\cos(2x-x)-\cos(2x+x)}{2}\right)^2 dx\\ &= \int \frac{1}{16}\left(\cos^2(x)-2\cos(x)\cos(3x)+\cos^2(3x)\right)dx\\ &= \int \frac{1}{16}\left(\frac{\cos(2x)+1}{2}-\cos(4x)-\cos(2x)+\frac{\cos(6x)+1}{2}\right)dx\\ &= \frac{1}{16}\left(\frac{-\sin(2x)}{2}-\frac{\sin(4x)}{4}+\frac{\sin(6x)}{12}+x\right)+C\end{aligned}$$

(d) Under the substitution $u=\sqrt{x}$ the integral becomes

$$\int \sqrt{1+\sqrt{x}}dx=2\int u\sqrt{u+1}du.$$

Therefore by integration by parts we get,

$$\int \sqrt{1+\sqrt{x}}dx=\frac{4}{3}u(u+1)^{3/2}-\int \frac{4}{3}(u+1)^{3/2}du=\frac{4}{15}(\sqrt{x}+1)^{3/2}(3\sqrt{x}-2).$$

(e) We have that $\dfrac{1}{\operatorname{sech}(x)} = \dfrac{2e^x}{e^{2x}-1}$ and so if we use the subsitution $u = e^x$, we get it into a form we can use a partial decomposition. The final answer will be $\ln\left|\dfrac{e^x+1}{e^x-1}\right| + C$.

(f) Using the substitution $x = \pi/2 - u$ we find that

$$\begin{aligned}
\int_0^{\pi/2} \frac{\sin(x)}{\cos(x)+\sin(x)}dx &= -\int_{\pi/2}^{0} \frac{\sin(\pi/2-u)}{\cos(\pi/2-u)+\sin(\pi/2-u)}du \\
&= \int_0^{\pi/2} \frac{\cos(x)}{\cos(x)+\sin(x)}dx.
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
2\int_0^{\pi/2} \frac{\sin(x)}{\cos(x)+\sin(x)}dx &= \int_0^{\pi/2} \frac{\sin(x)}{\cos(x)+\sin(x)}dx + \int_0^{\pi/2} \frac{\sin(x)}{\cos(x)+\sin(x)}dx \\
&= \int_0^{\pi/2} \frac{\sin(x)}{\cos(x)+\sin(x)}dx + \int_0^{\pi/2} \frac{\cos(x)}{\cos(x)+\sin(x)}dx \\
&= \int_0^{\pi/2} \frac{\cos(x)+\sin(x)}{\cos(x)+\sin(x)}dx \\
&= \frac{\pi}{2}.
\end{aligned}$$

Therefore, $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{\cos(x)+\sin(x)}dx = \frac{\pi}{4}$.

Question 6. Find the surface area of the torus obtained by rotating the circle $x^2 + (y-b)^2 = r^2$ about the x-axis.

Solution to Question 6.
The top semi circle is given by $y = b + \sqrt{r^2 - x^2}$ while the bottom semicircle is given by $y = b - \sqrt{r^2 - x^2}$. In both cases we have that $1 + (y')^2 = \frac{x^2}{\sqrt{r^2-x^2}}$. therefore, we conclude the total surface area is given by

$$\begin{aligned}
\text{SA} &= 2\pi \int_{-r}^{r} \left(b + \sqrt{r^2-x^2}\right) \frac{x^2}{\sqrt{r^2-x^2}}dx + 2\pi \int_{-r}^{r} \left(b - \sqrt{r^2-x^2}\right) \frac{x^2}{\sqrt{r^2-x^2}}dx \\
&= 4ab\pi \int_{-r}^{r} \frac{1}{\sqrt{r^2-x^2}}dx \\
&= 4ab\pi^2.
\end{aligned}$$