Math 31B: Week 3 Section

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Information

Discussion Questions

Question 1. Use L'Hopital's rule to evaluate the following limits or state that it does not apply

(a) $\displaystyle\lim_{x\to 9}\frac{x^{1/2}-x+6}{x^{3/2}-27}$

(b) $\displaystyle\lim_{x\to 0}\frac{\cos(2x)-1}{\sin(5x)}$

(c) $\displaystyle\lim_{x\to 0}\frac{\sin(4x)}{x^2+3x+1}$

(d) $\displaystyle\lim_{x\to 1}\frac{e^x-e}{2x-2}$

Solution to Question 1.

(a) We have $\displaystyle\lim_{x\to 9}\frac{x^{1/2}-x+6}{x^{3/2}-27}=\frac{0}{0}$ and so L-Hopital's applies. Differentiating we get that

$$\lim_{x\to 9}\frac{\frac{1}{2}x^{-1/2}-1}{\frac{3}{2}x^{1/2}}=\frac{-5}{27}.$$

Hence $\displaystyle\lim_{x\to 9}\frac{x^{1/2}+x-6}{x^{3/2}-27}=\frac{-5}{27}$.

(b) We have $\displaystyle\lim_{x\to 0}\frac{\cos(2x)-1}{\sin(5x)}=\frac{0}{0}$ and so L-Hopital's applies. Differentiating we get that

$$\lim_{x\to 0}\frac{-2\sin(2x)}{5\cos(5x)}=\frac{0}{5}.$$

Hence $\displaystyle\lim_{x\to 0}\frac{\cos(2x)-1}{\sin(5x)}=0$.

(c) We have $\displaystyle\lim_{x\to 0}\frac{\sin(4x)}{x^2+3x+1}=\frac{0}{1}$ and so L-Hopital's does not apply.

(d) We have $\displaystyle\lim_{x\to 1}\frac{e^x-e}{2x-2}=\frac{0}{0}$ and so L-Hopital's applies. Differentiating we get that

$$\lim_{x\to 1}\frac{e^x}{2}=\frac{e}{2}.$$

Hence $\displaystyle\lim_{x\to 1}\frac{e^x-e}{2x-2}=\frac{e}{2}$.

Question 2. Compute without calculator:

(a) $\arcsin(\sin\frac{\pi}{3})$

(b) $\arcsin(\sin\frac{4\pi}{3})$

(c) $\arctan(\tan\frac{3\pi}{4})$

(d) $\cos(\arctan(x))$

Solution to Question 2.

(a) Since the range of arcsin is $[\pi/2, \pi/2]$, we have that $\arcsin(\sin\frac{\pi}{3}) = \frac{\pi}{3}$.

(b) Since $\frac{4\pi}{3}$ is not in the range of arcsin, we can not use the previous solution. We want to find a value $y \in [\pi/2, \pi/2]$ such that $\sin(\frac{4\pi}{3}) = \sin(y)$, since this will then give us

$$\arcsin(\sin\frac{4\pi}{3}) = \arcsin(\sin y) = y.$$

In order to do this, we will use the identity $\sin(x) = \sin(\pi - x)$ (note, the general version of this identity is $\sin(x) = \sin((-1)^n(x - n\pi))$, $n \in \mathbb{Z}$ which can be used in the general case).

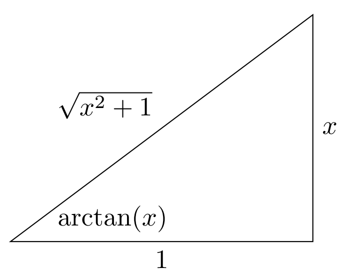
Now, we have that $\sin(\frac{4\pi}{3}) = \sin(\pi - \frac{4\pi}{3}) = \sin(-\frac{\pi}{3})$ and as $-\frac{\pi}{3}$ is in the range of arcsin we have that

$$\arcsin(\sin\frac{4\pi}{3}) = \arcsin(\sin\frac{-\pi}{3}) = \frac{-\pi}{3}.$$

(c) Similarly to the previous question. $\frac{3\pi}{4}$ is not in the range of arctan. However, we do have that $\tan(x)$ is π periodic ($\tan(x) = \tan(x + n\pi)$ for all $n \in Z$) and so

$$\arctan(\tan\frac{3\pi}{4}) = \arctan(\tan(\frac{3\pi}{4} - \pi)) = \arctan(\tan\frac{-\pi}{4}) = \frac{-\pi}{4}$$

(d) We use a triangle method. We have the triangle



Hence, we have that $\cos(\arctan(x)) = \dfrac{1}{\sqrt{x^2+1}}$.

Question 3. Show that $e = \lim_{x\to 0}(1 + x)^{1/x}$.

Solution to Question 3.
$\lim_{x\to 0}(1 + x)^{1/x} = 1^\infty$ which is an indeterminate form. We consider the logarithm,

$$\ln((1 + x)^{1/x}) = \frac{\ln(1 + x)}{x} \to \frac{0}{0} \text{ as } x \to 0.$$

Hence we try L'Hopitals and get that

$$\frac{(\ln(1 + x))'}{(x)'} = \frac{(1 + x)^{-1}}{1} \to 1 \text{ as } x \to 0.$$

Therefore, $\lim_{x\to 0}\ln(1 + x)^{1/x} = 1$ and as exponential is continuous we have that

$$\lim_{x\to 0}(1 + x)^{1/x} = \lim_{x\to 0}\exp(\ln(1 + x)^{1/x}) = \exp\left(\lim_{x\to 0}\ln(1 + x)^{1/x}\right) = \exp(1) = e.$$

Question 4. Find the derivatives of the following functions:

(a) $\arcsin(e^x)$

(b) $\arccos(\ln(x))$

(c) $\sec^{-1}(t+1)$

(d) $\tan^{-1}\left(\dfrac{1+t}{1-t}\right)$.

Solution to Question 4.

(a) $\dfrac{e^x}{\sqrt{1-e^{2x}}}$

(b) $\dfrac{-1}{x\sqrt{1-\ln^2(x))}}$

(c) $\dfrac{1}{|t+1|\sqrt{(t+1)^2-1}}$

(d) $\dfrac{1}{1+t^2}$.

Question 5. Evaluate the following integrals:

(a) $\displaystyle\int \frac{dt}{\sqrt{1-16t^2}}$

(b) $\displaystyle\int \frac{dx}{x\sqrt{x^4-1}}$

(c) $\displaystyle\int \frac{\ln(cos^{-1}(x))dx}{(\cos^{-1}(x))\sqrt{1-x^2}}$.

Solution to Question 5.

(a) $\dfrac{1}{4}\arcsin(4t)+C$. Use substitution $u=4t$.

(b) $\dfrac{1}{2}\sec^{-1}(x^2)+C$. Use substitution $u=x^2$.

(c) $-(\ln(\arccos(x)))^2+C$.
Use substitution $u=\arccos(x)$.

Homework Questions

Section 7.7
6, 10, 12, 16, 18, 26, 38, 44, 48, 50, 53, 54, 60, 62.
Section 7.8
30, 32, 34, 38, 48, 54, 56, 58, 62, 72, 112.

Extra Questions

Question 6. Show that 0^∞ is not an indeterminate form by showing that for any positive functions f and g such that $\lim_{x\to 0} f(x)=0$ and $\lim_{x\to 0} g(x)=\infty$, then

$$\lim_{x\to 0} f(x)^{g(x)} = 0.$$

In contrast, show that 1^∞ is an indeterminate form by finding an example of positive functions f,g such that $\lim_{x\to 0} f(x)=1$,$\lim_{x\to 0} g(x)=\infty$ and $\lim_{x\to 0} f(x)^{g(x)}=1$. And then find another pair of functions f,g with corresponding limits as $x\to\infty$ but $\lim_{x\to 0} f(x)^{g(x)}\neq 1$.

Solution to Question 6.
Consider the logarithm $\ln(f(x)^{g(x)}) = g(x)(x)$ which always exists since f is positive. By the limit laws we have

$$\lim_{x\to 0} g(x)\ln f(x) = \lim_{x\to 0} g(x) \cdot \lim_{x\to 0} \ln f(x) = \infty \times -\infty = -\infty.$$

Hence

$$\lim_{x\to 0} f(x)^{g(x)} = \lim_{x\to 0} \exp(\ln f(x)^{g(x)}) = \exp(\lim_{x\to 0} f(x)^{g(x)}) = 0.$$

Now consider the functions $f(x) = 1$ and $g(x) = 1/x$, then $\lim_{x\to 0} f(x) = 1$, $\lim_{x\to 0} g(x) = \infty$ and we have that

$$\lim_{x\to 0} f(x)^{g(x)} = 1.$$

If instead we take $f(x) = (1+x)$, then we still have that $\lim_{x\to 0} f(x) = 1$. However, from Question 3 we have that

$$\lim_{x\to 0} f(x)^{g(x)} = e.$$

Question 7. Evaluate the following integrals:

(a) $\displaystyle\int 2^x e^{4x} dx$

(b) $\displaystyle\int \frac{e^x dx}{\sqrt{1-16e^{2x}}}$

(c) $\displaystyle\int \cos(x)5^{-2\sin(x)} dx$

(d) $\displaystyle\int \frac{dx}{x\sqrt{25x^2-1}}$.

Solution to Question 7.

(a) $\dfrac{2^x e^{4x}}{4+ln2} + C$. Note $2^x e^4 x = e^{\ln(2)x+4x}$.

(b) $\dfrac{1}{4}\arcsin(4e^x) + C$. Use $u = 4e^x$.

(c) $\dfrac{-5^{-2\sin(x)}}{2\ln(5)} + C$. Use $u = \sin(x)$.

(d) $\sec^{-1}(5x) + C$. Use $u = 5x$.