Math 31B: Week 2 Section

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Information

My office hours are now: 2pm on Tue and 4pm on Thur.

Discussion Questions

Question 1. Solve the following equations:

(a) $7^{\log_7(21x)} = 3$

(b) $\ln(x^2 + 4) = 2\ln(x) + \ln(2)$

(c) $7e^{5t} = 100$

(d) $\log_3(y) + 3\log(y^2) = 14$

Solution to Question 1.

(a) $x = 1/7$

(b) $x = \pm 2$

(c) $t = \frac{1}{5}\ln\left(\frac{100}{7}\right)$

(d) $y = 9$

Question 2. find a domain on which f is one-to-one and a formula for the inverse of f restricted to this domain. Sketch the graphs of f and f^{-1}.

(a) $f(x) = \frac{1}{x+1}$

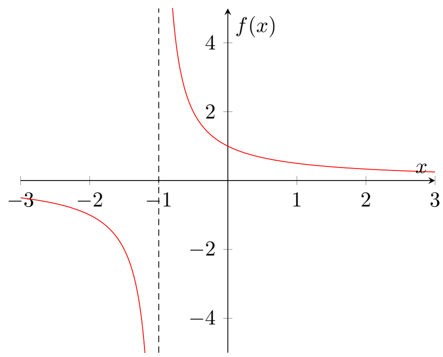
(b) $f(x) = \frac{1}{\sqrt{x^2+1}}$

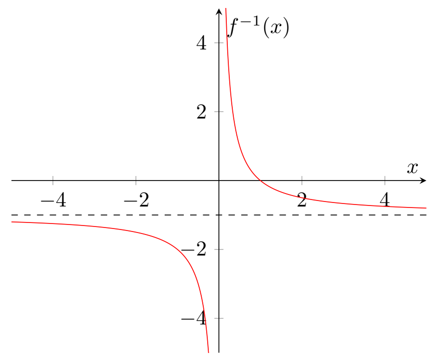
Solution to Question 2.

(a) The function f has a natural domain given by $x \neq -1$. The derivative is given by $f'(x) = -\frac{1}{(x+1)^2}$ and we see that $f' < 0$ when $x > -1$ and so f is decreasing in this region and therefore one-to-one for $x > -1$. Similarly, $f > 0$ for $x < -1$ and so is one-to-one for $x < -1$. Moreover, given that $f > 0$ for $x > -1$ and $f < 0$ for $x < -1$ we therefore conclude that f is one-to-one on the entire domain $x \neq -1$ and hence has a well defined inverse. Swapping y and x and solving for the inverse yields:

$$\begin{aligned} x &= \frac{1}{1+y} \\ 1 + y &= \frac{1}{x} \\ y &= \frac{1}{x} - 1 \end{aligned}$$

The graphs are given by the following:





(b) The natural domain for f is the entire real line. However, f is not one-to-one and we must restrict it in order to get an inverse. Notice that f is symmetric around the y-axis (i.e, an even function) and so we must atleast restrict f to either the positive or negative real numbers. Note, we may still need to restrict it more in order to ensure it is one-to-one.

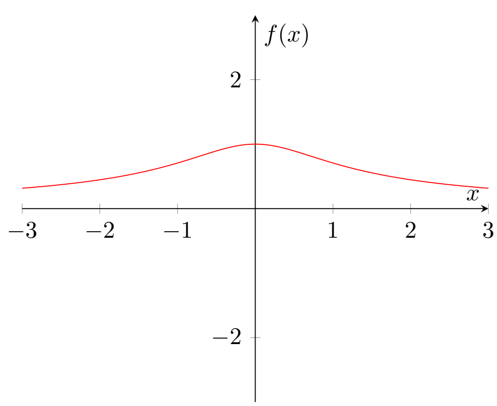
Differentiating f, we get that

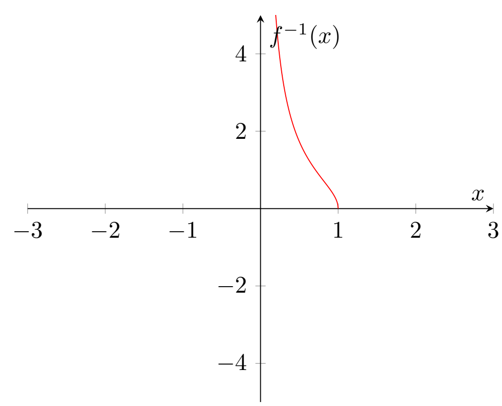
$$f'(x) = -\frac{x}{(x^2+1)^{3/2}}$$

and so we have that f is monotonically decreasing for $x > 0$ and and monotonically increasing for $x < 0$. Hence we can restrict f to the domain $x > 0$, for which it is one-to-one and so has an inverse. Now, solving for the inverse we swap the x and y. Noting that since we have restricted to the domain $x > 0$, the inverse has it range in the region $y > 0$.

$$\begin{aligned} x &= \frac{1}{\sqrt{y^2+1}} \\ x^2 &= \frac{1}{y^2+1} \\ y^2 &= \frac{1}{x^2} - 1 \\ \therefore y &= \sqrt{\frac{1}{x^2} - 1} \text{ since } y > 0 \text{ i.e, take positive root.} \end{aligned}$$

The graphs are given by the following:





Question 3. We have from lectures that if g is the inverse for a differentiable and one-to-one function f, then for x with $x \neq 0$,

$$g'(x) = \frac{1}{f'(g(x))}.$$

(a) Let $f(x) = x^3 + 1$ and g it's inverse. Find a formula for $g(x)$ and calculate g' in two ways. The first by differentiating g, and the second way by applying the above theorem.

(b) Let $f(x) = x^3 + 2x + 4$ and g it's inverse. Without finding a formula for $g(x)$ (no seriously, don't even try) calculate $g(7)$ and then $g'(7)$.

Solution to Question 3.

(a) Swapping the x and y, we then solve $x = y^3 + 1$ which gives us that the inverse is $y = (x-1)^{1/3}$. i.e, $g(x) = (x-1)^{1/3}$.

Differentiating, we get $g'(x) = \frac{1}{3}(x-1)^{-2/3}$. Alternatively, $f(x) = 3x^2$ and so we get that

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{3(x-1)^{2/3}}.$$

(b) $g(7) = b$ where b is the number such that $f(b) = 7$. i.e, $b^3 + 2b + 4 = 7$ and so $b = 1$ by inspection. Now, by the above theorem we have that

$$g'(7) = \frac{1}{f'(g(7))} = \frac{1}{f'(1)} = \frac{1}{5}.$$

Question 4. Calculate the following derivatives

(a) $y = \ln(x^2 6^x)$

(b) $y = \ln\left(\frac{x+1}{x^3+1}\right)$

(c) $y = 8^{\cos(x)}$

(d) $y = x^{e^x}$

Solution to Question 4.

(a)

$$\begin{aligned} y &= \ln(x^2 6^x) \\ y &= 2\ln(x) + x\ln(6) \\ \frac{dy}{dx} &= \frac{2}{x} + \ln(6). \end{aligned}$$

(b)

$$\begin{aligned} y &= \ln\left(\frac{x+1}{x^3+1}\right) \\ y &= \ln(x+1) - \ln(x^3+1) \\ \frac{dy}{dx} &= \frac{1}{x+1} - \frac{3x^2}{x^3+1}. \end{aligned}$$

(c)

$$\begin{aligned} y &= 8^{\cos(x)} \\ y &= e^{\ln(8)\cos(x)} \\ \frac{dy}{dx} &= -\ln(8)\sin(x)e^{\ln(8)\cos(x)} \\ &= -\ln(8)\sin(x)8^{\cos(x)}. \end{aligned}$$

(d)

$$\begin{aligned} y &= x^{e^x} \\ y &= e^{\ln(x)e^x} \\ \frac{dy}{dx} &= \frac{d}{dx}(\ln(x)e^x) \cdot e^{\ln(x)e^x} \\ &= \left(\frac{e^x}{x} + \ln(x)e^x\right) \cdot x^{e^x}. \end{aligned}$$

Homework Questions

Section 7.2

16, 18, 20, 22, 26, 32, 36

Section 7.3

30, 34, 38, 46, 48, 76, 80

Extra Questions

Question 5. Differentiate the following:

(a) $y = \frac{x(x^2+1)}{\sqrt{x+1}}$

(b) $y = x^{3^x}$

(c) $y = \pi^{5x-2}$

(d) $y = (2x+1)(4x^2)\sqrt{x-9}$

Solution to Question 5.

(a)

$$\begin{aligned} y &= \frac{x(x^2+1)}{\sqrt{x+1}} \\ \ln(y) &= \ln(x) + \ln(x^2+1) - \frac{1}{2}\ln(x+1) \\ \frac{1}{y}\frac{dy}{dx} &= \frac{1}{x} + \frac{2x}{x^2+1} - \frac{1}{2(x+1)} \\ \therefore \frac{dy}{dx} &= \frac{x^2+1}{\sqrt{x+1}} + \frac{2x^2}{\sqrt{x+1}} - \frac{x(x^2+1)}{2(x+1)^{3/2}}. \end{aligned}$$

(b)

$$\begin{aligned} y &= x^{3^x} \\ y &= e^{\ln(x)3^x} \\ \frac{dy}{dx} &= \frac{d}{dx}(\ln(x)3^x) \cdot e^{\ln(x)3^x} \\ &= \left(\frac{3^x}{x} + \ln(x)\ln(3)e^x\right) \cdot x^{3^x}. \end{aligned}$$

(c)

$$\begin{aligned} y &= \pi^{5x-2} \\ y &= e^{\ln(\pi)(5x-2)} \\ \frac{dy}{dx} &= 5\ln(\pi)e^{\ln(\pi)(5x-2)} \\ &= 5\ln(\pi)\pi^{5x-2}. \end{aligned}$$

(d)

$$\begin{aligned} y &= (2x+1)(4x^2)\sqrt{x-9} \\ \ln(y) &= \ln(2x+1) + \ln(4) + 2\ln(x) + \frac{1}{2}\ln(x-9) \\ \frac{1}{y}\frac{dy}{dx} &= \frac{1}{2x+1} + \frac{2}{x} + \frac{1}{2(x-9)} \\ \therefore \frac{dy}{dx} &= 8x^2\sqrt{x-9} + 8x(2x+1)\sqrt{x-9} + \frac{(2x+1)(2x^2)}{\sqrt{x-9}}. \end{aligned}$$

Ofcourse, we could have just as easily have gotten this from the product rule.

*** Question 6.** Prove the formula $\log_a(b)\log_b(a) = 1$ for all positive numbers a, b with $a \neq 1$ and $b \neq 1$.

Solution to Question 6.
Since $a \neq 1$ and $b \neq 1$, neither $log_a(b)$ or $\log_b(a)$ are zero. Hence,

$$\log_a(b)\log_b(a) = \log_b(a^{\log_b(a)}) = \log_b(b) = 1.$$

*** Question 7.** Let f be a differentiable function with inverse g such that $f(x) = f'(x)$. Show that $g'(x) = x^{-1}$.

Solution to Question 7.
We have that

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \text{ when } f'(g(x)) \neq 0 \\ &= \frac{1}{f(g(x))} \text{ since } f = f' \\ &= \frac{1}{x} \text{ since } g \text{ inverse to } f. \end{aligned}$$

Note that $f'(g(x)) = 0 \iff f(g(x)) = 0 \iff x = 0$. Hence the above formula works for all $x \neq 0$.