Math 31B: Week 10 Section

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Discussion Questions

Question 1. Find the interval of convergence for the following

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{\ln(n)}$

(b) $\displaystyle\sum_{n=1}^{\infty} n(x-3)^n$

Solution to Question 1.

(a) Using the ratio test we find that $\dfrac{|a_{n+1}|}{|a_n|} = \left|\dfrac{x^{n+1}}{\ln(n+1)}\right| \cdot \left|\dfrac{\ln(n)}{x^n}\right| \to |x|$. Hence the power series converges absolutely for $|x| < 1$. Now we check the end points.

When $x = 1$, we compare with the harmonic series to see that it divereges. When $x = -1$ we can apply the alternating series test and we see that it converges. Hence the series converges on the interval $[-1, 1)$.

(b) Using the ratio test we see that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n+1}{n}|x-3| \to |x-3| \text{ as } n \to \infty.$$

Hence the series converges absolutely for $|x-3| < 1$. We now check the end points. When $x = \pm 4$, this diverges by the n-th term divergence test. Hence we see that the interval of convergence is $(2, 4)$.

Question 2. We have that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1.$$

Use this and the equality $\dfrac{1}{1-x} = \dfrac{-1}{1+(x-2)}$ to show that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n \text{ for } |x-2| < 1.$$

Solution to Question 2.
We have that

$$\begin{aligned}
\frac{1}{1-x} &= \frac{-1}{1+(x-2)} \\
&= -\sum_{n=0}^{\infty} (-(x-2))^n \text{ for } |x-2| < 1 \\
&= -\sum_{n=0}^{\infty} (-1)^n (x-2)^n \\
&= \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n
\end{aligned}$$

Question 3. Find The following Maclaurin series and the interval the expansion is valid by using previously known series.

(a) $f(x) = \dfrac{1-\cos(x)}{x}$

(b) $f(x) = (x^2+1)\sin(x)$

Solution to Question 3.

(a) We know that $\cos(x) = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n}}{(2n)!}$ for all x. Hence it follows for all x that

$$\begin{aligned}\frac{1-\cos(x)}{x} &= \frac{1-\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}{x}\\ &= \frac{\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}{x}\\ &= \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n)!}\end{aligned}$$

(b) Similarly, we know that $\sin(x) = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$ for all x. Hence we have that

$$\begin{aligned}(x^2+1)\sin(x) &= (x^2+1)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\\ &= x^2\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+3}}{(2n+1)!} + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\\ &= x + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+3}}{(2n+1)!} + \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\\ &= x + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+3}}{(2n+1)!} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+3}}{(2n+3)!}\\ &= x + \sum_{n=0}^{\infty}\left(1-\frac{1}{(2n+3)(2n+2)}\right)(-1)^n\frac{x^{2n+3}}{(2n+1)!}\\ &= x + \sum_{n=0}^{\infty}(-1)^n\frac{(4n^2+10n+5)x^{2n+3}}{(2n+3)!}.\end{aligned}$$

Question 4. Show that

$$\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots$$

converges to zero. How many terms must be computed to get within 0.01 of zero?

Solution to Question 4.

We have that $\sin(x) = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{(2n+1)!}$ for all x and so we see this series converges to $\sin(\pi) = 0$. Power

series coincide with their taylor expansion and so we can use the error estimate for the taylor polynomial of $\sin(x)$ around $x = 0$ to understand how many terms we need to compute the series to get it within 0.01 of zero. i.e, the first N terms of the series are exactly $T_{2N-1}(\pi)$.
We find that

$$|\sin(\pi) - T_{2N-1}(\pi)| \leq \max_{x \in [0,\pi]} \frac{|\sin^{(2N)}(x)|\pi^{2N}}{(2N)!} \leq \frac{\pi^{2N}}{(2N)!}$$

We want this less than 10^2 and so we see that $N = 10$ is enough after putting this into a calculator.