Math 31B: Mock Midterm 2

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Time: 40 minutes. Note, this practice midterm is shorter than the other practice midterm to reflect the shorter time. In particular, question 3 is significantly shorter.

Question 1.

(a) Compute S_6 for the integral $\int_0^1 x dx$.

(b) The error bound for the trapezoidal rule approximation to $\int_a^b f(x)dx$ is given by

$$\left| \int_a^b f(x)dx - T_N \right| \leq \max_{x\in[a,b]} \frac{|f''(x)||b-a|^3}{12N^2}.$$

If $f(x) = e^{-x}$ and $[a,b] = [0,3]$, what should N be if the right hand side of the error bound is to be less than or equal 10^{-6}?

Solution to Question 1.

(a) By the formula we have that

$$\begin{aligned} S_6 &= \frac{1}{3}\frac{1}{6}\left(0 + 1 + 2(\frac{2}{6} + \frac{4}{6}) + 4(\frac{1}{6} + \frac{3}{6} + \frac{5}{6})\right) \\ &= \frac{1}{2}. \end{aligned}$$

(b) We have that $\max_{x\in[0,3]} |f''(x)| = 1$ since the exponential is decreasing. Hence by the above formula, we want N such that

$$\frac{3^3}{12N^2} \leq 10^{-6} \implies N \geq \frac{3}{2} \times 10^3 = 1500.$$

Question 2.

(a) Calculate the arclength of $y = 9 - 3x$ over the interval $[1,3]$.

(b) Calculate the surface of revolution around the x-axis of $y = \sin(x)$ over the interval $[0,\pi]$. You may use

$$\int \sqrt{1+u^2}du = \frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln(u + \sqrt{1+u^2}) + C.$$

Solution to Question 2.

(a) We have $\sqrt{1+(y')^2} = \sqrt{10}$. Hence we have that

$$\text{Arclength} = \int_1^3 \sqrt{1+(y')^2}dx = 2\sqrt{10}.$$

(b) We have that

$$\begin{aligned}\text{Surface area} &= 2\pi \int_0^{\pi} \sin(x)\sqrt{1+\cos^2(x)}dx \\ &= 2\pi \int_{-1}^{1} \sqrt{1+u^2}du \text{ after substitution } u=\cos(x) \\ &= 2\pi \left(\sqrt{2}+\frac{1}{2}\ln(1+\sqrt{2})-\frac{1}{2}\ln(-1+\sqrt{2})\right) \\ &= 2\pi\left(\sqrt{2}+\ln(\sqrt{2}+1)\right).\end{aligned}$$

Question 3.

(a) Calculate the third Maclaurin polynomial of $\arcsin(x)$.

Solution to Question 3.

(a) You should get

$$T_3(x) = x + \frac{1}{6}x^3.$$

Question 4.

(a) Evaluate $\displaystyle\int_{\infty}^{\infty} \frac{1}{x^2+1}dx$.

(b) Use the comparison test to prove that $\displaystyle\int_1^{\infty} \frac{dx}{x^2+\sinh(x)}$ converges.

Solution to Question 4.

(a) We have that

$$\begin{aligned}\int_{\infty}^{\infty} \frac{1}{x^2+1}dx &= \int_0^{\infty}\frac{1}{x^2+1}dx + \int_{-\infty}^{0}\frac{1}{x^2+1}dx \\ &= \lim_{a\to\infty}\int_0^{a}\frac{1}{x^2+1}dx + \lim_{a\to-\infty}\int_a^{0}\frac{1}{x^2+1}dx \\ &= \lim_{a\to\infty}\arctan(a) - \lim_{a\to-\infty}\arctan(a) \\ &= \pi.\end{aligned}$$

(b) Since $\sinh(x) > 0$ for $x \geq 1$, we have that

$$0 \leq \frac{1}{x^2+\sinh(x)} \leq \frac{1}{x^2}.$$

Since $\displaystyle\int_1^{\infty}\frac{dx}{x^2}$ converges as it's a p-integral, by the comparison test we have that $\displaystyle\int_1^{\infty}\frac{dx}{x^2+\sinh(x)}$ converges.