Mock Midterm 1 solutions

Q1

(a) $$\frac{d}{dx}(x^2e^{2x}) = 2xe^{2x} + 2x^2e^{2x}$$
$$= 2x(x+1)e^{2x}$$

(b) $$\frac{d}{dx}(e^{\sin x}) = \cos(x)e^{\sin x}$$

(c) $$\frac{d}{dx}(\tan(e^{5-6x})) = \sec^2(e^{5-6x})\cdot\frac{d}{dx}(e^{5-6x})$$
$$= -6e^{5-6x}\sec^2(e^{5-6x})$$

(d) $$\frac{d}{dx}(e^{1/x}) = e^{1/x}\frac{d}{dx}(1/x)$$
$$= -\frac{e^{1/x}}{x^2}$$

(e) $$\frac{d}{dx}(4^{-2x}) = \frac{d}{dx}(e^{-2\ln(4)x})$$
$$= -2\ln(4)e^{-2\ln(4)x}$$
$$= -2\ln(4)4^{-2x}$$

Q2

(a) $\frac{d}{dx}(x\ln(x)-x) = \ln(x) + \frac{x}{x} - 1$

$= \ln(x)$

(b) $\frac{d}{dx}\left(\ln((\ln x)^3)\right) = \frac{d}{dx}(3\ln(\ln x))$

$= \frac{1}{\ln x} \cdot \frac{d}{dx}(3\ln x)$

$= \frac{3}{x\ln x}$

Alternatively,

$\frac{d}{dx}\left(\ln((\ln x)^3)\right) = \frac{1}{(\ln x)^3} \cdot 3(\ln x)^2 \cdot \frac{1}{x}$

$= \frac{3}{x\ln x}.$

(c) $\frac{d}{dx}\ln\left(\frac{x+1}{x^3+1}\right) = \frac{d}{dx}\ln(x+1) - \frac{d}{dx}\ln(x^3+1)$

$= \frac{1}{x+1} - \frac{3x^2}{x^3+1}$

- Rule of thumb: $+$ and $-$ are easier to deal with than $\times$ and $\div$, so use log to change them when possible.

ie, if we didn't use the property $\ln(a/b) = \ln(a) - \ln(b)$ in last question

we get that

$$\frac{d}{dx} \ln\left(\frac{x+1}{x^3+1}\right) = \frac{1}{x+1/x^3+1} \cdot \frac{(x+1)'(x^3+1) - (x^3+1)'(x+1)}{(x^3+1)^2}$$

$$= \frac{x^3+1 - (x+1)\cdot 3x^2}{(x+1)(x^3+1)}$$

$$= \frac{1 - 2x^3}{(x+1)(x^3+1)}$$

which requires more work.

(d) the complicated powers suggests logarithmic differentiation over a straight application of the quotient rule.

(d) continued:

$$y = \frac{(x+12)^{5/2}}{(x-6)^{1/5}}$$

$$\ln y = \frac{5}{2}\ln(x+12) - \frac{1}{5}\ln(x-6)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{5}{2(x+12)} - \frac{1}{5\ln(x-6)}$$

$$\therefore \frac{dy}{dx} = \left(\frac{5}{2(x+12)} - \frac{1}{5\cancel{\ln}(x-6)}\right)\frac{(x+12)^{5/2}}{(x-6)^{1/5}}$$

Q3

(1) we have that $f'(x) = \ln b \cdot b^x$.

Since $1<b$, $\ln b>0$ and as $b^x>0$

we conclude that $f'(x)>0$ and so

f is strictly increasing.

(2) Since $\ln(x)$, $x>0$ is continuous we have

$$\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right) = \ln\left(\lim_{x\to\infty} 1+\frac{1}{x}\right)$$

$$= \ln(1)$$

$$= 0$$

(3) we have by the change of base formula

$$f(x) = \log_b(x) = \frac{\ln(x)}{\ln(b)}$$

and so $f'(x) = \dfrac{1}{x\ln(b)}$.

as $x>0$, $0<b<1 \Rightarrow \ln(b)<0$ we see

that $f'(x)<0$ and so f is decreasing.

(4) let $u = e^t + 1$, then $du = e^t dt$ and we have

$$\int e^t \sqrt{e^t + 1} = \int \sqrt{u}\, du$$

$$= \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{3} (e^t + 1)^{3/2} + C$$

(5) let $u = \ln(t)$, then $du = \frac{1}{t} dt$.

when $t = e$, $u = 1$
$t = e^2$, $u = 2$

Hence, after u-substitution we get

$$\int_e^{e^2} \frac{1}{t \ln(t)} dt = \int_1^2 \frac{du}{u} = \ln(u) \Big|_1^2 = \ln(2)$$

Q4

(1) we check that $g(f(x)) = x$ on domain of f

and $f(g(x)) = x$ on domain of g.

$$g(f(x)) = \frac{1 - \frac{1}{1+x}}{\frac{1}{1+x}} = \frac{1+x-1}{1} = x$$

$$f(g(x)) = \frac{1}{1 + \frac{1-x}{x}} = \frac{x}{x+1-x} = x$$

Hence f and g are inverses.

(2) $$g'(b) = \frac{1}{f'(g(b))}$$

(3) by inspection, $f(1) = 0$ and so $f^{-1}(0) = 1$

Now, by the above formula, we have

$$f^{-1}(0) = \frac{1}{f'(f^{-1}(0))} = \left.\frac{e^x}{(e^x - e)^2 + 1}\right|_{x=1}$$

$$= e$$

(4) $\ln f(x) = x\ln(2) + 2x\ln(3) + x^2\ln(e)$

differentiating,

$$\frac{f'(x)}{f(x)} = \ln(2) + 2\ln(3) + 2x$$

$$\therefore f'(x) = (\ln(2) + 2\ln(3) + 2x)\, 2^x \cdot 3^{2x} \cdot e^{x^2}$$

$$= (\ln(18) + 2x)\, 2^x \cdot 3^{2x} \cdot e^{x^2}$$

Note, if the question didn't specify logarithmic differentiation, we could do the following;

$$f(x) = 2^x \cdot 3^{2x} \cdot e^{x^2}$$

$$= e^{\ln(2)x} \cdot e^{2x\ln(3)} \cdot e^{x^2}$$

$$= e^{\ln(2)x + 2x\ln(3) + x^2}$$

$$\therefore f'(x) = (\ln(2) + 2\ln(3) + 2x) \cdot 2^x \cdot 3^{2x} \cdot e^{x^2}.$$

Q5

(i) $$\lim_{x \to 0} \left(\frac{1}{\sin(x)} - \frac{1}{x} \right) = \lim_{x \to 0} \left(\frac{x - \sin(x)}{x \sin(x)} \right)$$

$$= \frac{0}{0}$$ which is indeterminate.

Hence, l'hopitals gives

$$LHS = \lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x) + x\cos(x)} = \frac{0}{0}$$

L'hopitals again gives

$$LHS = \lim_{x \to 0} \frac{\sin(x)}{2\cos(x) - x\sin(x)} = \frac{0}{2}$$

Hence, $$\lim_{x \to 0} \left(\frac{1}{\sin(x)} - \frac{1}{x} \right) = 0.$$

Alternate method without L'hopitals:

We already know the limit $\lim_{x \to 0} \frac{x}{\sin(x)} = 1$.

Consider,

$$\lim_{x \to 0} \frac{x - \sin(x)}{x \sin(x)} \cdot \frac{\sin x}{x} = \lim_{x \to 0} \frac{x - \sin(x)}{x^2}$$

$$= \lim_{x \to 0} \frac{1}{x} - \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \frac{1}{x}$$ by limit laws

$$= 0 - 1 \cdot 0 = 0.$$

Now,

$$\lim_{x \to 0} \frac{x - \sin(x)}{x \sin(x)} \cdot \frac{\sin(x)}{x} \cdot \frac{x}{\sin(x)}$$

$$= \lim_{x \to 0} \frac{x - \sin(x)}{x \sin(x)} \cdot \frac{\sin(x)}{x} \cdot \lim_{x \to 0} \frac{x}{\sin(x)}$$ by limit laws

$$= 0 \cdot 1 = 0$$

(2) $\lim_{x \to 1} \frac{x(\ln x - 1) + 1}{(x-1)\ln x} = \frac{0}{0}$ which is indeterminate.

by l'hopitals

$$\lim_{x \to 1} \frac{x(\ln x - 1) + 1}{(x-1)\ln x} = \lim_{x \to 1} \frac{\ln x - 1 + \frac{x}{x}}{\frac{x-1}{x} + \ln x}$$

$= \frac{0}{0}$ which is again indeterminate so by L'Hopitals,

$$LHS = \lim_{x\to 1} \frac{1/x}{1/x^2 + 1/x} = \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2}$$

Hence $\displaystyle\lim_{x\to 1} \frac{x(\ln x - 1)}{(x-1)\ln x} = \frac{1}{2}$

(3) $\displaystyle\lim_{x\to 0} \frac{\cos(x+\pi/2)}{\sin(x)} = \frac{0}{0}$ which is indeterminate

by L'Hopitals,

$$LHS = \lim_{x\to 0} \frac{-\sin(x+\pi/2)}{\cos(x)} = -\frac{1}{1} = -1.$$

Hence
$$\lim_{x\to 0} \frac{\cos(x+\pi/2)}{\sin(x)} = -1.$$

Q6

(1) $f(x) = e^{\arccos(x)}$

$$f'(x) = \frac{d}{dx}(\arccos(x))\, e^{\arccos(x)}$$

$$= \frac{-e^{\arccos(x)}}{\sqrt{1-x^2}}$$

(2) $$\int_0^3 \frac{dx}{x^2+3} = \frac{1}{3}\int_0^3 \frac{dx}{\frac{x^2}{3}+1}$$

let $u = \frac{x}{\sqrt{3}}$, then $du = \frac{1}{\sqrt{3}}dx$

when $x=0$, $u=0$

$x=3$, $u=\sqrt{3}$

Hence LHS $= \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} \frac{du}{u^2+1}$

$$= \frac{1}{\sqrt{3}} \arctan(x)\Big|_0^{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \arctan(\sqrt{3})$$

$$= \frac{\pi}{3\sqrt{3}}$$

3) we have $\sinh(x) \to \infty$ as $x \to \infty$

and $\tanh(x) \to 1$ as $x \to \infty$.

putting these together gives

$$\lim_{x \to \infty} \tanh(\sinh(x)) = \tanh\left(\lim_{x \to \infty} \sinh(x)\right)$$
$$= \infty.$$

Note, it is useful to know the graphs of the hyperbolic functions.

(4) $\cosh^{-1}(x)$ is the inverse of $\cosh(x)$ for $x \geq 0$.

from the previous question, we know that

$$\frac{d}{dx} \cosh^{-1}(x) = \frac{1}{\cosh'(\cosh^{-1}(x))}$$ for when x in domain of $\cosh^{-1}(x)$ $(x > 1)$

now, the derivative of cosh is sinh, hence

$$\frac{d}{dx} \cosh^{-1}(x) = \frac{1}{\sinh(\cosh^{-1}(x))}$$

$$= \frac{1}{\sqrt{\cosh(\cosh^{-1}(x))^2 - 1}} \quad \text{as } \cosh^2(x) - \sinh^2(x) = 1$$

$$= \frac{1}{\sqrt{x^2-1}}$$

Alternatively,

$y = \cosh^{-1}(x) \Rightarrow \cosh(y) = x$ and implicitly differentiating gives

$$\sinh(y) \cdot \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sinh(y)} = \frac{1}{\sqrt{\cosh^2(y) - 1}} = \frac{1}{\sqrt{x^2-1}}$$

as $x = \cosh(y)$.

Q7

(1) $\int \arcsin^{-1}(x) \overset{IBP}{=} x\arcsin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$

let $u = 1-x^2$, $du = -2x\,dx$

then $LHS = x\arcsin^{-1}(x) + \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$= x\arcsin^{-1}(x) + u^{1/2} + c$

$= x\arcsin^{-1}(x) + (1-x^2)^{1/2} + c$

(2) $\int_0^1 xe^{-x} \overset{IBP}{=} -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx$

$= e^{-1} - e^{-x}\Big|_0^1$

$= e^{-1} - e^{-1} + 1$

$= 1$

Q8

(1) $$\frac{3x^2+5x-4}{(x-2)(x+1)^2} = \frac{A}{x-2} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

and so ☆

$$3x^2+5x-4 = A(x+1)^2 + B(x-2)(x+1) + C(x-2)$$

The general method is to equate the coefficients and then solve the system of 3 equations in 3 unknowns. However, we can use a few tricks here.

When $x=2$, we have (☆) is:

$$3(2)^2+5(2)-4 = 3^2A$$
$$12+10-4 = 9A \Rightarrow A = 2$$

Equating the coeffecent of x^2 in ☆ gives

$$3 = A+B \Rightarrow B=1$$

when $x=-1$ in ☆, we get

$$3(-1)^2-5-4 = -3C$$
$$-6 = -3C \Rightarrow C=2.$$

Hence, $A=2$, $B=1$, $C=2$.

(2) we have

$$\int \frac{2x^2-2x+4}{(x-1)(x^2+1)}\,dx = \int \frac{2}{x-1} - \frac{2}{x^2+1}\,dx$$

$$= 2\ln(x-1) - 2\arctan(x) + C.$$