Question 1

(a) we have the inequalities:

$$10 \le a_n = (n + 10^n)^{1/n} \le (10^n + 10^n)^{1/n}$$
$$= (2 \cdot 10^n)^{1/n}$$
$$= 2^{1/n} \cdot 10.$$

since $2^{1/n} \to 1$ as $n \to \infty$, by squeeze thm we have

$$\lim_{n \to \infty} a_n = 10$$

(b) $a_n = \ln(n^2+1) - \ln(n^2-1)$

$$= \ln\left(\frac{n^2+1}{n^2-1}\right)$$

$$= \ln\left(\frac{1 + 1/n^2}{1 - 1/n^2}\right)$$

since $\dfrac{1+1/n^2}{1-1/n^2} \to 1$ as $n \to \infty$

and $\ln(x)$ continuous for $x>0$, we get

$$\lim_{n\to\infty} a_n = \lim_{x\to 1} \ln(x) = 0.$$

Question 2

(a) Let $a_n = \frac{e^n}{n^n}$, we have that

$$\sqrt[n]{a_n} = \frac{e}{n} \to 0 \quad \text{as} \quad n\to\infty$$

Hence by root test, this converges absolutely.

(b) Let $a_n = (-1)^n n^2$. Since $\lim_{n\to\infty} a_n \neq 0$ (ie $\lim_{n\to\infty} |a_n| = \infty$) by the nth term divergence test, the series $\sum a_n$ diverges.

(c) let $a_n = \frac{1}{\sqrt{n}+\ln(n)}$, $b_n = \frac{1}{\sqrt{n}}$ (note $a_n \geq 0, b_n \geq 0$)

$$\frac{a}{b_n} = \frac{\sqrt{n}}{\sqrt{n}+\ln(n)} = \frac{1}{1+\frac{\ln(n)}{\sqrt{n}}} \to 1 \quad \text{as } n\to\infty$$

Hence by the limit comparison test

$\sum a_n$ converges $\Leftrightarrow$ $\sum b_n$ converges.

But $\sum b_n$ is a p-series with $p = \frac{1}{2}$.

Hence $\sum \frac{1}{\sqrt{n} + \ln(n)}$ diverges.

(d) Observe that $\cos(n\pi) = (-1)^n$ and

so $$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\ln(n)} = \sum_{n=1}^{\infty} (-1)^n a_n$$

where $a_n = \frac{1}{\ln(n)}$. Now, a_n is positive, decreasing and $a_n \to 0$ as $n \to \infty$. Hence alternating series test applies and we conclude it converges.

The series doesn't converge absolutely.

Since $\frac{1}{\ln(n)} \geq \frac{1}{n}$, we see by direct comparison that as

$$\left| \frac{\cos(n\pi)}{\ln(n)} \right| = \frac{1}{\ln(n)} \geq \frac{1}{n}$$

and $\sum \frac{1}{n}$ harmonic series, so diverges that $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\ln(n)}$ doesn't converge absolutely.

(e) Since $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$, we have that

$\lim_{n \to \infty} \frac{\sin(1/n)}{1/n} = 1$. Now, let $a_n = \frac{\sin(1/n)}{\sqrt{n}}$

and $b_n = \frac{1}{n^{3/2}}$. We have

$\frac{a_n}{b_n} = \frac{\sin(1/n)}{1/n} \to 1$ as $n \to \infty$.

Hence by ~~the~~ limit comparison test (both series are positive) $\sum \frac{\sin(1/n)}{\sqrt{n}}$ converges $\iff$ $\sum 1/n^{3/2}$ converges. But $\sum 1/n^{3/2}$ is a p-series with $p = 3/2$ and so converges. Hence $\sum \frac{\sin(1/n)}{\sqrt{n}}$ converges

It also converges absolutely since its a positive series.

Question 3

(a) Let $a_n = \dfrac{x^{2n+1}}{3n+1}$

Then $\left|\dfrac{a_{n+1}}{a_n}\right| = |x|^2 \dfrac{3n+4}{3n+1} \to |x|^2$ as $n \to \infty$.

Hence by ratio test, $\sum_{n=0}^{\infty} a_n$ converges absolutely for $|x|<1$ and diverges for $|x|>1$.

Now we test end points.

When $x=1$,

$$\frac{a_n}{1/n} = \frac{1}{3+1/n} \to \frac{1}{3} \quad \text{as} \quad n \to \infty$$

Hence by limit comparison, this diverges as $\sum 1/n$ diverges (harmonic when $x=-1$, then we have

$$a_n = \frac{(-1)^{2n+1}}{3n+1} = \frac{-1}{3n+1}$$ and so also

diverges as it's the negative of the $x=1$ case.

Hence the interval of convergence is $(-1,1)$.

(b) Let $a_n = e^n (x-2)^n$

$$\sqrt[n]{|a_n|} = e|x-2|$$

and so by root test the series $\sum a_n$

converges absolutely for $|x-2| < 1/e$

and diverges for $|x-2| > 1/e$.

Now we test end points

when $x = 2 + 1/e$, $a_n = 1$.

Hence $\sum a_n$ diverges.

when $x = 2 - 1/e$ we have $a_n = (-1)^n$

and so also diverges.

Hence the interval of convergence is

$$\left(2-e^{-1},\ 2+e^{-1}\right)$$

Question 4

From the binomial theorem, we have for $|x|<1$

$$(1-x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-x)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1/2)(-1/2-1)\cdots(-1/2-n+1)}{(1)\ (2)\ \cdots\ (n)} (-x)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)(-3)(-5)\cdots(-2n+1)}{(2)(4)(6)\cdots(2n)} (-x)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{(2)(4)(6)\cdots(2n)} x^n$$

so for $|x|<1$ we have

$$(1-x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} x^{2n}$$

we can integrate term by term inside radius of convergence, this then gives us

$$\sin^{-1}(x) = C + x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{x^{2n+1}}{2n+1}$$

Since ~~s(0)~~ $\sin^{-1}(0) = 0 \Rightarrow C = 0$ and we are done. $\boxtimes$