# Week 6 Notes

**THEOREM 2 Error Bound** Assume that $f^{(n+1)}$ exists and is continuous. Let $K$ be a number such that $|f^{(n+1)}(u)| \le K$ for all $u$ between $a$ and $x$. Then

$$\boxed{|f(x) - T_n(x)| \le K\frac{|x-a|^{n+1}}{(n+1)!}}$$

where $T_n$ is the $n$th Taylor polynomial centered at $x = a$.

Example: Given $f(x) = \sqrt{1+x}$, find an error bound for $|\sqrt{8.2} - T_3(8.2)|$ when $T_3$ centered at 8.

step ①: Find K. Note, to use formula, $a = 8$ and $x = 8.2$ in this example. So we want to find K such that $\max_{u\in[8,8.2]} |f^{(4)}(u)| \le K$.

$$f(x) = \sqrt{1+x}$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$$

$$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$$

$$f^{(4)}(x) = -\frac{15}{\phantom{16}}(1+x)^{-7/2}.$$

$\frac{\quad}{16}$

So $|f^{(4)}(x)| = \frac{15}{16}(1+x)^{-7/2}$ which is decreasing between 8, 8.2. Hence take $K = |f^{(4)}(8)|$

$= \frac{15}{16} 5^{-7/2}$

(2) Plug into Formula:

$$|f(8.2) - T_3(8.2)| \leq \frac{K\,|8.2-8|^4}{4!}$$

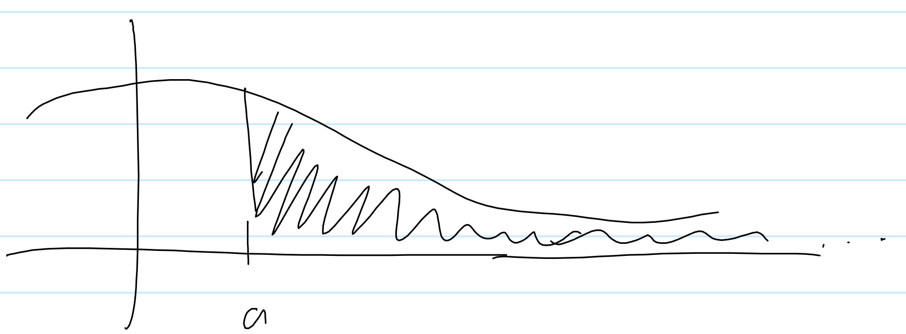
$$= \frac{K \times 0.2^4}{4!}$$

## Improper Integrals.

Two types

(1) infinite intervals:

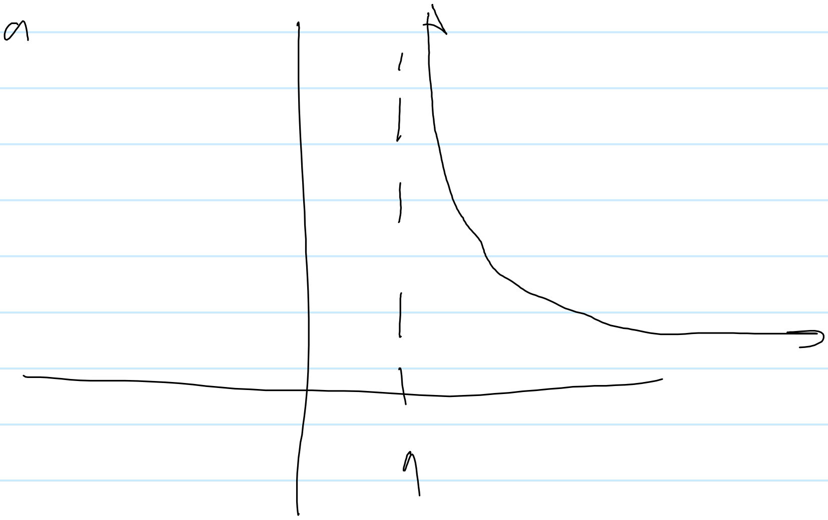
$$\int_{-\infty}^{a} f(x)\,dx \qquad \int_{a}^{\infty} f(x)\,dx$$



a

(2) The function is infinite at some point we are integrating over:

ie, $\int_a^b f(x)\,dx$



These kinds of integrals are defined in terms of limits. The idea is you make the problem point a variable and solve the integral, then take the limit as the variable goes to that problem point. If the limit is finite, it converges, if it doesn't exist or is $\infty$, the integral diverges.

Example: does $\int_0^1 \frac{1}{x^{1/2}}\,dx$ converge/diverge?

If it converges, what does it converge to?

Solution: This is the second type, the integrand $\to \infty$ as $x \to 0$. We replace the problem point with variable $a$. and solve.

$$\int_a^1 \frac{1}{x^{1/2}}\,dx = 2\sqrt{x}\,\Big|_a^1 = 2 - 2\sqrt{a}$$

and take the limit as $a \to 0$.

Hence $\displaystyle\int_0^1 \frac{1}{x^{1/2}}\,dx = \lim_{a\to 0} \int_a^1 \frac{1}{x^{1/2}}\,dx$

$$= 2$$

Hence this converges (to 2).

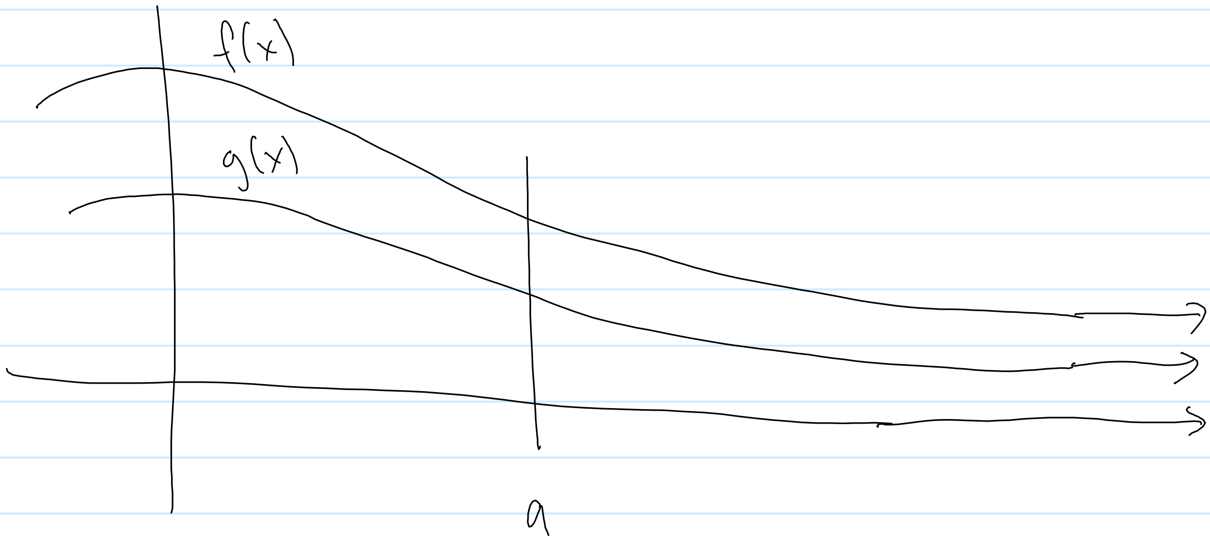
It is not always possible to solve the integral, and so we can't always find an exact value. However we can use other methods to tell if the improper integral converges/diverges.

**THEOREM 3 Comparison Test for Improper Integrals**
Assume that $f(x) \geq g(x) \geq 0$ for $x \geq a$:

- If $\displaystyle\int_a^\infty f(x)\,dx$ converges, then $\displaystyle\int_a^\infty g(x)\,dx$ also converges.
- If $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ also diverges.

The Comparison Test is also valid for improper integrals with infinite discontinuities at the endpoints.

The direction is important here! (Also, that these are positive functions)




Think in terms of Area: if $\int_a^\infty f(x)\,dx$ converges, then this has finite area under it, so it follows that $g(x)$ also has finite area. ie $\int_a^\infty g(x)\,dx$ converges.

while if $\int_a^\infty g(x)\,dx < \infty$ ie $g(x)$ has finite area, this doesn't imply $f(x)$ has finite area under it! ($\int f(x)\,dx < \infty$).

<u>Example:</u> Does $\int_1^\infty \frac{1}{x^4+e^x}\,dx$ converge or diverge?

<u>Solution:</u> We have that

$0 \leq \frac{1}{\quad} < 1$ for $x \geq 1$

$$0 \leq \frac{1}{x^4+e^x} \leq \frac{1}{e^x} \quad \text{for } x \geq 1$$

Hence if $\int_1^\infty \frac{1}{e^x}dx$ converges, then $\int_1^\infty \frac{1}{x^4+e^x}dx$ converges by comparison.

Now, $\int_1^\infty \frac{1}{e^x}dx = \lim_{b\to\infty} \int_1^b e^{-x}dx$

$$= \lim_{b\to\infty} -e^{-x}\Big|_1^b$$

$$= \lim_{b\to\infty} e^{-1} - e^{-b}$$

$$= e^{-1} < \infty.$$

Hence this converges and hence so does $\int_1^\infty \frac{dx}{e^x+x^4}$.

Note: when doing these kinds of questions, make sure you have the correct inequality and mention in your answer you are using the comparison test.

Note about arclengths.

**THEOREM 1 Formula for Arc Length** Assume that $f'$ exists and is continuous on the interval $[a, b]$. Then the arc length $s$ of $y = f(x)$ over $[a, b]$ is equal to

$$\boxed{s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx} \qquad \boxed{1}$$

These kinds of questions are usually pretty straight-forward. I just want to mention it is very common for these questions to be designed in such a way that

$1 + (f'(x))^2$ can be rewritten as a square, and you should always try to do this before integrating

e.g. the example from the book:

**EXAMPLE 1** Find the arc length $s$ of the graph of $f(x) = \frac{1}{12}x^3 + x^{-1}$ over the interval $[1, 3]$ (Figure 4).

**Solution** First, let's calculate $1 + f'(x)^2$. Since $f'(x) = \frac{1}{4}x^2 - x^{-2}$,

$$1 + f'(x)^2 = 1 + \left(\frac{1}{4}x^2 - x^{-2}\right)^2 = 1 + \left(\frac{1}{16}x^4 - \frac{1}{2} + x^{-4}\right)$$
$$= \frac{1}{16}x^4 + \frac{1}{2} + x^{-4} = \left(\frac{1}{4}x^2 + x^{-2}\right)^2$$

Fortunately, $1 + f'(x)^2$ is a square, so we can easily compute the arc length:

$$s = \int_1^3 \sqrt{1 + f'(x)^2}\,dx = \int_1^3 \left(\frac{1}{4}x^2 + x^{-2}\right) dx = \left(\frac{1}{12}x^3 - x^{-1}\right)\Big|_1^3$$
$$= \left(\frac{9}{4} - \frac{1}{3}\right) - \left(\frac{1}{12} - 1\right) = \frac{17}{6}$$