Week 2 Notes

From textbook:

THEOREM 2 Derivative of the Inverse Assume that f is differentiable and one-to-one with inverse $g(x) = f^{-1}(x)$. If b belongs to the domain of g and $f'(g(b)) \neq 0$, then $g'(b)$ exists and

$$\boxed{g'(b) = \frac{1}{f'(g(b))}} \qquad \boxed{3}$$

Example: Let $f(x) = \sqrt{3-x}$, $x \leq 3$. Let g be the inverse function of f. Find $g(2)$ and $g'(2)$.

Answer 1 (Textbook answer)

Let $b = g(2)$, this is the number such that $f(b) = 2$ by definition. ie

$$\sqrt{3-b} = 2$$

$$3 - b = 4$$

$$\therefore b = -1.$$

Hence we have $g(2) = -1$.

Now, $f'(x) = \dfrac{-1}{2\sqrt{3-x}}$ and so we have that

$$f'(g(2)) = f'(-1) = \frac{-1}{2 \times \sqrt{4}} = -\frac{1}{4}.$$

Hence by above formula $g'(2) = -4$.

Answer 2. (A bit more old school I suppose)

Some background: Notice that the above formula can be rewritten in terms of Liebnitz notation as
$$g'(b) = \frac{1}{\left.\frac{dy}{dx}\right|_{(g(b))}}$$
where $y = f(x)$. In otherwords you can think of $g'(b)$ as "$\left.\frac{dx}{dy}\right|_b$". This agrees with thinking of the inverse function as flipping the role of x and y. So we can do this problem anotherway:

we set
$$y = f(x) = \sqrt{3-x}$$
$$y^2 = 3 - x$$
differentiating gives
$$2y\frac{dy}{dx} = -1$$
$$\Rightarrow \frac{dy}{dx} = \frac{-1}{2y}$$

$$\Rightarrow \frac{dx}{dy} = -2y$$ (This is somewhat informal, I really mean $1/\frac{dy}{dx}$)

Now, we want the derivative of the inverse when $y=2$, so $g'(2) = \left.\frac{dx}{dy}\right|_{y=2} = -4$

Example: Find $\int \frac{(\ln x)^2}{x}\, dx$

We use u-substitution, let $u = \ln x$, then $du = \frac{1}{x} dx$ and so $\int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{u^3}{3} + C$

substituting back gives

$$\int \frac{(\ln x)^2}{x} dx = \frac{(\ln x)^3}{3} + C$$

Example: Differentiate $f(x) = \frac{x(x+1)^3}{(3x-1)^2}$

We will use a trick called logarithmic differentiation.

Note: Whenever you see something that has alot of multiplication, you should immediately think of using logarithms since they turn multiplication (hard) into addition (easy)

We have $\ln f(x) = \ln(x) + 3\ln(x+1) - 2\ln(3x-1)$ by log rules. Differentiating gives

$$\frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{3}{x+1} - \frac{6}{3x-1}$$

$$\Rightarrow f'(x) = f(x)\left(\frac{1}{x} + \frac{3}{x+1} - \frac{6}{3x-1}\right)$$

$$= \frac{x(x+1)^3}{(3x-1)^2}\left(\frac{1}{x} + \frac{3}{x+1} - \frac{6}{3x-1}\right)$$