Midterm II (A)

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This test totals 40 points and you get 45 minutes to do it. Answer the questions in the spaces provided on the question sheets. Show work unless the question says otherwise. Good luck!

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Discussion section :

Question	Points	Score
1	17	
2	6	
3	7	
4	6	
5	4	
Total:	40	

1. (a) (10 points) Find the following integrals. Simplify as much as possible and show your work to get full credit.

$$\int_1^\infty \frac{\ln x}{x^4}\, dx$$

$$\int_1^\infty \frac{\ln x}{x^4}\, dx = \lim_{R\to\infty} \int_1^R \frac{\ln x}{x^4}\, dx$$

$$(\text{IBP}) \quad = \lim_{R\to\infty} -\frac{\ln x}{3x^3}\Big|_1^R + \frac{1}{3}\int_1^R \frac{dx}{x^4}$$

$$= \lim_{R\to\infty} -\frac{\ln R}{3R^3} - \frac{1}{3^2}\cdot\frac{1}{x^3}\Big|_1^R$$

$$= \lim_{R\to\infty} -\frac{\ln R}{3R^3} - \frac{1}{9R^3} + \frac{1}{9}$$

$$= \frac{1}{9}$$

by elementary limits $\lim_{R\to\infty} \frac{\ln R}{R} = 0$ and $\lim_{R\to\infty} \frac{1}{R} = 0$.

(b) (7 points)

$$\int \frac{x^3}{x-1}\,dx$$

We have $\dfrac{x^3}{x-1} = \dfrac{x^3-1}{x-1} + \dfrac{1}{x-1}$

$$= \frac{(x-1)(x^2+x+1)}{x-1} + \frac{1}{x-1}$$

$$= x^2+x+1 + \frac{1}{x-1}$$

Hence, $\displaystyle\int \frac{x^3}{x-1}\,dx = \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1| + C$.

Alternatively, by long division we have

$$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x-1 & x^3 \\ & \underline{x^3 - x^2} \\ & \quad +x^2 \\ & \quad \underline{x^2 - x} \\ & \qquad x \\ & \qquad \underline{x - 1} \\ & \qquad\quad 1 \end{array}$$

$\Rightarrow \quad x^3 = (x^2+x+1)(x-1) + 1.$

2. (6 points) Find $T_3(x)$, the third Taylor polynomial of $f(x) = x^2 + 1$ centered at $a = 2$. Also find the maximum possible value of the error $|f(2.1) - T_3(2.1)|$. Show your work to get full credit.

$f(x) = x^2 + 1 \qquad f(2) = 5$

$f'(x) = 2x \qquad f'(2) = 4$

$f''(x) = 2 \qquad f''(2) = 2$

$f'''(x) = 0 \qquad f'''(2) = 0.$

Now, $T_3(x) = f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$

$\therefore T_3(x) = 5 + 4(x-2) + (x-2)^2.$

The error bound is given by

$|f(2.1) - T_3(2.1)| \leq \frac{K|2.1-2|^4}{4!}$ where $K \geq |f^{(4)}(u)|$ for all $u \in [2, 2.1]$

Since $f^{(4)}(u) = 0$ in this case, we can take $K = 0$.

Hence $|f(2.1) - T_3(2.1)| = 0$, ie no error.

Aside: Polynomials agree with their taylor polys of similar degree. So in this case we have $f(x) = T_3(x)$ (can see this by simplifying above). Hence there is no error since we have equality.

3. (7 points) Does the following integral converge or diverge ? Explain your answer carefully with proper reasoning, stating which theorems you use.

$$\int_4^\infty \frac{dx}{\sqrt{x}+\cos^2 x}$$

Since $\cos^2(x) \le 1 \le \sqrt{x}$ for all $x \geq 4$, we have that $\sqrt{x} + \cos^2(x) \le \sqrt{x} + \sqrt{x} = 2\sqrt{x}$ for $x \geq 4$ and so

$$\frac{1}{\sqrt{x}+\cos^2 x} \geq \frac{1}{2\sqrt{x}} \geq 0.$$

The comparison test now applies, if $\int_4^\infty \frac{1}{2\sqrt{x}}dx$ diverges, then $\int_4^\infty \frac{dx}{\sqrt{x}+\cos^2 x}$ diverges.

However, we can see $\int_4^\infty \frac{1}{2\sqrt{x}}dx$ diverges a) it is a p-integral or alternatively;

$$\int_4^\infty \frac{1}{2\sqrt{x}}dx = \lim_{R\to\infty} \int_4^R \frac{dx}{2\sqrt{x}} = \lim_{R\to\infty} \sqrt{x}\Big|_4^R = \infty.$$

Therefore, we conclude $\int_4^\infty \frac{dx}{\sqrt{x}+\cos^2 x}$ diverges.

4. (6 points) Find the surface area of the figure obtained by rotating $y = \sqrt{16 - x^2}$ about the x-axis over the interval $[-2, 2]$. Show neat work to get full credit.

$$y = \sqrt{16 - x^2}$$

$$y' = \frac{1}{2\sqrt{16-x^2}} \cdot -2x = \frac{-x}{\sqrt{16-x^2}}$$

$$1 + (y')^2 = 1 + \frac{x^2}{16-x^2} = \frac{16}{16-x^2}$$

Now,

$$SA = 2\pi \int_{-2}^{2} f(x)\sqrt{1 + (f'(x))^2}\,dx$$

$$= 2\pi \int_{-2}^{2} \sqrt{16-x^2}\sqrt{\frac{16}{16-x^2}}\,dx$$

$$= 2\pi \int_{-2}^{2} 4\,dx$$

$$= 8\pi\, x\Big|_{-2}^{2} = 32\pi$$

5. (4 points) Find the limit of the following sequences. (Write if DNE limit does not exist). You don't have to show work for this question and no partial credit will be given.

   (a) $a_n = \cos n$, find $\lim_{n\to\infty} a_n$.

   DNE

   (b) $a_n = \frac{\cos n}{n}$, find $\lim_{n\to\infty} a_n$.

   $\lim_{n \to \infty} a_n = 0.$

Cheat sheet

List of derivatives of inverse trignometric and inverse hyperbolic functions

1. $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{x^2+1}$
4. $\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{x^2+1}$
5. $\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$
6. $\frac{d}{dx}\text{cosec}^{-1}(x) = \frac{-1}{|x|\sqrt{x^2-1}}$
7. $\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{x^2+1}}$
8. $\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$
9. $\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$
10. $\frac{d}{dx}\coth^{-1}(x) = \frac{1}{1-x^2}$
11. $\frac{d}{dx}\text{sech}^{-1}(x) = \frac{-1}{x\sqrt{1-x^2}}$
12. $\frac{d}{dx}\text{cosech}^{-1}(x) = \frac{-1}{|x|\sqrt{x^2+1}}$

p-test

Let $a > 0$. Then,

1. $\int_a^\infty \frac{dx}{x^p}$ converges for $p > 1$ and diverges for $p \leq 1$.
2. $\int_0^a \frac{dx}{x^p}$ converges for $p < 1$ and diverges for $p \geq 1$.

Space for scratch work. (WILL NOT BE EVALUATED)
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