MATH 31B: Week 8

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Question 1. Use the definition of series to find the value of the series $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ if it converges. Hint: partial fractions.

Solution to Question 1.
We have by partial decomposition that $\frac{1}{4n^2-1} = \frac{1/2}{2n-1} - \frac{1/2}{2n+1}$. Hence we get via telescoping series that the partial sums are

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{4k^2-1} \\ &= \sum_{k=1}^{n} \frac{1/2}{2k-1} - \frac{1/2}{2k+1} \\ &= 1/2 - \frac{1/2}{2n+1}. \end{aligned}$$

Hence, we have

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1/2 - \frac{1/2}{2n+1} = 1/2.$$

Question 2. Which of the following inequalities can be used to study the convergence of $\sum_{n=1} \frac{1}{n^2+\sqrt{n}}$?

$$\frac{1}{n^2+\sqrt{n}} \le \frac{1}{n^2} \quad \text{or} \quad \frac{1}{n^2+\sqrt{n}} \le \frac{1}{\sqrt{n}}$$

Explain and then determine whether the series converges or not.

Solution to Question 2.
Since both of the sequences are greater than the given one, we need the larger one to give a convergent series in order to use the direct comparison test. Since the left one converges while the right one diverges (p-series) we pick the left inequality and conclude our series converges.

Question 3. Use the limit comparison test to determines whether the series $\sum_{n=1}^{\infty} \frac{2n^3+3n}{n^6+n^5+\sqrt{n}}$ converges or diverges.

Solution to Question 3.
Let $a_n = \frac{2n^3+3n}{n^6+n^5+\sqrt{n}}$ and $b_n = \frac{1}{n^3}$. A quick calculation shows that $\lim_{n\to\infty} \frac{a_n}{b_n} = 2$. Since $\sum b_n$ converges as it's a p-series with $p = 2$, we conclude by limit comparison test that $\sum_{n=1}^{\infty} \frac{2n^3+3n}{n^6+n^5+\sqrt{n}}$ converges.

Question 4. Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - \sqrt{n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n + n}{e^{2n} - \sqrt{n}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n! + 4^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^3}$

Solution to Question 4.
The following a short answers. Actual answers should be closer to the previous question. You should always state exactly what test you are using in your answer.

(a) Converges. Limit comparison test with $\dfrac{1}{n^2}$

(b) Converges. Either direct comparison test with $\dfrac{1}{n2^n} \leq \dfrac{1}{2^n}$ or limit comparison test with $\dfrac{1}{2^n}$.

(c) Converges. Limit comparison test with $\dfrac{1}{e^n}$ (this is geometric).

(d) Converges. Direct or limit comparison test with $\frac{1}{4^{n-1}}$.

(e) Diverges. n-th term divergence test.