MATH 31B: Week 6 Discussion

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Question 1. Determine whether the following improper integrals converge or diverge. If they converge, what do they converge to?

(a) $\int_1^\infty x^3 dx$

(b) $\int_0^2 x^2 dx$

(c) $\int_1^2 \frac{1}{x\ln(x)} dx$

Solution to Question 1.

(a)

$$\begin{aligned}\int_1^\infty x^3 dx &= \lim_{R\to\infty} \int_1^R x^3 dx \\ &= \lim_{R\to\infty} \left.\frac{x^4}{4}\right|_1^R \\ &= \lim_{R\to\infty} \frac{R^4}{4} - \frac{1}{4} \\ &= \infty.\end{aligned}$$

Hence this diverges

(b)

$$\begin{aligned}\int_0^2 x^2 dx &= \lim_{r\to 0} \int_r^2 x^2 dx \\ &= \lim_{r\to 0} \left.\frac{x^3}{3}\right|_r^2 \\ &= \lim_{r\to 0} \frac{8}{3} - \frac{r^3}{3} \\ &= \frac{8}{3}.\end{aligned}$$

Hence converges to 8/3.

(c)

$$\begin{aligned}\int_1^2 \frac{1}{x\ln(x)} dx &= \lim_{r\to 1} \int_r^2 \frac{1}{x\ln(x)} dx \\ &= \lim_{r\to 1} \ln(\ln(x))|_r^2 \\ &= \lim_{r\to 0} \ln(\ln(2)) - \ln(\ln(r)) \\ &= \infty.\end{aligned}$$

Hence this diverges.

Question 2. Use the comparison test to determine whether the following integrals converge or diverge.

(a) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^4+3}}$

(b) $\displaystyle\int_0^1 \frac{dx}{x^4+\sqrt{x}}$

Solution to Question 2.

(a) We have the inequality $\sqrt{x^4+3} > \sqrt{x^4} = x^2$ for $x > 1$. Hence $0 \leq \dfrac{1}{\sqrt{x^4+3}} \leq \dfrac{1}{x_2}$. Since $\displaystyle\int_1^\infty \frac{1}{x^2}dx$ converges (either by above method or as a p-integral), the comparison test implies that $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^4+3}}$ converges.

(b) We have the inequality $x^4 + \sqrt{x} \geq \sqrt{x}$ for $x \in [0,1]$. Hence $0 \leq \dfrac{1}{x^4+\sqrt{x}} \leq \dfrac{1}{\sqrt{x}}$. Since $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}dx$ converges (either by above method or as a p-integral), the comparison test implies that $\displaystyle\int_0^1 \frac{dx}{x^4+\sqrt{x}}$ converges.

Question 3. Compute the arclength of $y = \left(\dfrac{x}{2}\right)^4 + \dfrac{1}{2x^2}$ over the interval $[1,4]$.

Solution to Question 3.
We have that $y' = \dfrac{x^3}{4} - \dfrac{1}{x^3}$ and so

$$\begin{aligned} 1+(y')^2 &= 1 + \left(\frac{x^3}{4} - \frac{1}{x^3}\right)^2 \\ &= 1 + \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6} \\ &= \frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6} \\ &= \left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2. \end{aligned}$$

Hence arclength is given by

$$\begin{aligned} \text{Arclength} &= \int_1^4 \sqrt{1+(y')^2}dx \\ &= \int_1^4 \frac{x^3}{4} + \frac{1}{x^3}dx \\ &= \frac{509}{32} \end{aligned}$$

Question 4. (More Challenging) Compute the arc length of $y = \ln\left(\dfrac{e^x+1}{e^x-1}\right)$ over the interval $[1,3]$.

Solution to Question 4.
We have the following:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-2e^x}{e^{2x}-1} \\
\left(\frac{dy}{dx}\right)^2 &= \frac{4e^{2x}}{(e^{2x}-1)^2} \\
1+\left(\frac{dy}{dx}\right)^2 &= \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2} \\
\sqrt{1+\left(\frac{dy}{dx}\right)^2} &= \frac{e^{2x}+1}{e^{2x}-1} \\
&= \coth(x) \text{ for } x \geq 0.
\end{aligned}$$

Hence we have that

$$\text{Arclength} = \int_1^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^3 \coth(x)dx = \ln(\sinh(3)) - \ln(\sinh(1)).$$