MATH31B: Week 5

TA: Ben Szczesny | Last updated: 2018/05/07

Question 1. Write the following in summation notation

(a) $\frac{1}{2 \cdot 3} + \frac{2}{3.4} + \cdots + \frac{n}{(n+1)(n+2)}$

(b) $3x^2 + 4x^3 + \cdots 30x^{29}$

(c) $7x^6 + 9x^8 + 11x^{10} + \cdots 31x^{30}$

Solution to Question 1.

(a) $\sum_{i=1}^{n} \frac{i}{(i+1)(i+2)}$

(b) $\sum_{i=2}^{29} (i+1)x^i$

(c) $\sum_{i=3}^{15} (2i+1)x^{2i}$

Question 2. Find the $2n$-th degree Taylor polynomial of $\cos(x)$ around the point $a = 0$ and write it in summation notation.

Solution to Question 2.
We calculate the derivatives at zero and look for a pattern.

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \\ f''(0) &= -1 \\ f^{(3)}(0) &= 0 \\ f^{(4)}(0) &= 1. \end{aligned}$$

At this point, we notice that the pattern will always repeat after here. From here we can see that the terms in the taylor polynomial with odd derivatives will be zero and we only take the even terms.
We then get

$$T_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!}$$

In summation notation, this is $\sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{(2i)!}$.

Question 3. For the following functions f, find a value K such that for the given n and interval, $|f^{(n)}(x)| \leq K$ for all x in that interval.

(a) $f(x) = x^4$, $n = 3$ on $[0, 1]$

(b) $f(x) = \dfrac{1}{x}$, $n = 4$ on $[1, 2]$

(c) $f(x) = \cos(x)$ for all n and all $x \in \mathbb{R}$.

Solution to Question 3.

(a) Differentiating gives $|f^{(3)}(x)| = |24x| = 24x$ on $[0, 1]$. Since this is an increasing function, it achieves it's maximum over this interval at $x = 1$. Hence we can take $K = f^{(3)}(1) = 24$.

(b) Differentiating gives $|f^{(4)}(x)| = |\frac{24}{x^5}| = \frac{24}{x^5}$ on $[1, 2]$. This function is decreasing so achieves it's maximum at $x = 1$. Hence we can take $K = f^{(4)}(1) = 24$.

(c) We have that $|f^{(n)}(x)| = |\cos(x)|$ or $|\sin(x)|$, both of which are always bounded above by 1. Hence we can just take $K = 1$ in this case.

Question 4. Use the error bound for the Taylor polynomial to find error bounds for the following:

(a) $|f(0.1) - T_7(0.1)|$ where $f(x) = e^x$ and T_7 is centred at $a = 0$.

(b) $|f(4.3) - T_2(4.3)|$ where $f(x) = x^{-1/2}$ and T_2 is centred at $a = 4$.

Solution to Question 4.

(a) We have that $f^{(8)}(x) = e^x$ in this case. Since this is an increasing function, we get that it's maximum over the interval $[0, 0.1]$ occurs at $x = 0.1$. Hence we can take $K = e^{0.1}$. Therefore the error is bounded by

$$|f(0.1) - T_7(0.1)| \leq \frac{e^{0.1}0.1^8}{8!}.$$

(b) We have $f^{(3)}(x) = \frac{-15}{8}x^{-7/2}$. For $x \in [4, 4.3]$ we have that $|f^{(3)}(x)| = \left|\frac{-15}{8}x^{-7/2}\right| = \frac{15}{8}x^{-7/2}$. This is a decreasing function and so max is achieve at $x = 4$. Hence we can take $K = \frac{15}{8}4^{-7/2} = \frac{15}{8 \times 2^7} = 15 \times 2^{-10}$.We then have the error bounded as follows:

$$|f(4.3) - T_2(4.3)| \leq \frac{15 \times 2^{-10} \times 0.3^3}{3!}.$$

Question 5. Use the error bound for the taylor polynomial to find a value for n such that $|\cos(0.1) - T_n(0.1)| \leq 10^{-7}$ holds. Here T_n is centred at $a = 0$.

Solution to Question 5.
K is a number larger than the maximum of the absolute value of the $n + 1$-th derivative over the interval $[0, 0.1]$. Since $|\cos(x)|$ and $|\sin(x)|$ are bounded above by 1 we can just take $K = 1$. Substituting our values into the inequality gives

$$|T_n(0.1) - \cos(0.1)| \leq \frac{1}{10^{n+1}(n+1)!}.$$

We want this less than 10^{-7}. i.e,

$$\frac{1}{10^{n+1}(n+1)!} \leq 10^{-7}.$$

Rearranging gives us that

$$(n + 1)! \geq 10^{6-n}$$

which by inspection we can see that we can take $n = 6$ or greater. Ofcourse, we can also see that n=4,5 also work after a bit of calculation.

Question 6. Evaluate $\int \frac{18}{(x+3)(x^2+9)} dx$.

Solution to Question 6.
$x^2 + 9$ is irreducible. Hence we have a decomposition of the form

$$\frac{18}{(x+3)(x^2+9)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+9}$$
$$18 = A(x^2+9) + (Bx+C)(x+3)$$

Equating coefficients gives us

$$A + B = 0$$
$$3B + C = 0$$
$$9A + 3C = 18.$$

Solving gives $A = 1, B = -1, C = 3$. Hence we get

$$\int \frac{18}{(x+3)(x^2+9)} dx = \int \frac{1}{x+1} dx - \int \frac{x}{x^2+9} dx + \int \frac{3}{x^2+9} dx$$
$$= \ln|x+3| = \frac{1}{2} \ln|x^2+9| + \arctan\left(\frac{x}{3}\right) + C.$$