MATH31B: Week 3 Mock Midterm

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Question 1. Show that $f(x) = \dfrac{1}{x^2+1}$ is one-to-one on $(-\infty, 0]$ and find a formula for f^{-1} for this domain of f.

Solution to Question 1.
Differentiating gives $f'(x) = \dfrac{-2x}{(x^2+1)^2} > 0$ for $x \in (-\infty, 0)$. Hence we conclude f is strictly increasing on $(-\infty, 0]$ and hence one-to-one.
Now, let $y = f(x)$. Solving for x gives

$$\begin{aligned} y &= \frac{1}{x^2+1} \\ \frac{1}{y} &= x^2 + 1 \\ \frac{1}{y} - 1 &= x^2 \\ \therefore x &= -\sqrt{\frac{1}{y} - 1}. \end{aligned}$$

Note that we take the negative root as $x \in (-\infty, 0]$. Hence we find the inverse (after swapping x and y) to be

$$f^{-1}(x) = -\sqrt{\frac{1}{x} - 1}$$

Question 2. Given that $1 - \tanh^2(x) = \operatorname{sech}^2(t)$, prove that $\dfrac{d}{dx} \tanh^{-1}(x) = \dfrac{1}{1-x^2}$.

Solution to Question 2.
Let $f(x) = \tanh(x)$. Then we have that $f'(x) = \operatorname{sech}^2(x)$ and by the derivative of the inverse formula, we get that

$$\frac{d}{dx} \tanh^{-1}(x) = \frac{1}{\operatorname{sech}^2(\tanh^{-1}(x))} = \frac{1}{1 - \tanh^2(\tanh^{-1}(x))} = \frac{1}{1-x^2}$$

since they are inverses.

Question 3. Evaluate $\displaystyle\lim_{x \to 2} \frac{e^{x^2} - e^4}{x - 2}$.

Solution to Question 3.
Substituting $x = 2$ gives us $0/0$, an indeterminant form. Hence L'hopitals applies. We then have

$$\begin{aligned} \lim_{x \to 2} \frac{e^{x^2} - e^4}{x - 2} &= \lim_{x \to 2} \frac{2xe^{x^2}}{1} \\ &= 4e^4. \end{aligned}$$

Question 4. Differentiate

(a) $y = (2x+1)(4x^2)\sqrt{x-9}$

(b) $y = \ln(\arcsin(x))$

Solution to Question 4.

(a) We have
$$\ln(y) = \ln(2x+1) + \ln(4) + 2\ln(x) + \frac{1}{2}\ln(x-9).$$
Differentiating yields
$$\frac{y'}{y} = \frac{2}{2x+1} + \frac{2}{x} + \frac{1}{2(x-9)}.$$
Hence
$$y' = (2x+1)(4x^2)\sqrt{x-9}\left(\frac{2}{2x+1} + \frac{2}{x} + \frac{1}{2(x-9)}\right)$$

(b) By the chain rule, we have that
$$\frac{dy}{dx} = \frac{1}{\arcsin(x)\sqrt{1-x^2}}$$

Question 5. Evaluate the following integrals

(a) $\displaystyle\int \frac{dx}{\sqrt{1-16x^2}}$

(b) $\displaystyle\int 3^x dx$

(c) $\displaystyle\int e^x \cos(x) dx$

Solution to Question 5.

(a) let $u = 4x$, so $du = 4dx$. Hence
$$\int \frac{dx}{\sqrt{1-16x^2}} = \frac{1}{4}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{4}\arcsin(u) + C = \frac{1}{4}\arcsin(4x) + C$$

(b) We have
$$\int 3^x dx = \int e^{\ln(3)x} dx = \frac{e^{\ln(3)x}}{\ln(3)} + C = \frac{3^x}{\ln(3)} + C$$

(c) Let $I = \displaystyle\int e^x \cos(x) dx$. Doing integration by parts twice, we get that
$$\begin{aligned} I &= \int e^x \cos(x) dx \\ &= e^x \cos(x) + \int e^x \sin(x) dx \\ &= e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x) dx \\ &= e^x(\cos(x) + \sin(x)) - I. \end{aligned}$$

Rearranging gives $I = \dfrac{e^x}{2}(\cos(x) + \sin(x)) + C$.