MATH31B: Week 2 discussion

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Discussion Questions

Question 1. Let $f(x) = x + \cos(x)$ and $g(x)$ it's inverse. Calculate $g(1)$ and $g'(1)$.

Solution to Question 1.
Let $b = g(1)$, this is a number such that $f(b) = 1$ since it's an inverse. i.e,

$$b + \cos(b) = 1.$$

By inspection, we find that $b = 0$ and so $g(1) = 0$. Now, we will use the formula for the derivative of the inverse to find $g'(1)$. We first differentiate f, $f'(x) = 1 - \sin(x)$. Hence we have

$$\begin{aligned} g'(1) &= \frac{1}{f'(g(1))} \\ &= \frac{1}{f'(0)} \\ &= 1. \end{aligned}$$

Question 2. Calculate $\int \frac{\cos(x)}{2\sin(x)+3}dx$. Hint: u-substitution.

Solution to Question 2.

$$\begin{aligned} \int \int \frac{\cos(x)}{2\sin(x)+3}dx &= \int \frac{1}{2u+3}du \text{ after } u = \sin(x) \\ &= \frac{1}{2}\ln|2u+3| + C \\ &= \frac{1}{2}\ln|2\sin(x)+3| + C \end{aligned}$$

Question 3. Find the derivative of $f(x) = \frac{x(x^2+1)}{\sqrt{x+1}}$.

Solution to Question 3.
We have that

$$\ln(f(x)) = \ln(x) + \ln(x^2+1) - \frac{1}{2}\ln(x+1).$$

Differentiating gives us

$$\frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{2x}{x^2+1} - \frac{1}{2(x+1)}.$$

Hence

$$f'(x) = \frac{x(x^2+1)}{\sqrt{x+1}}\left(\frac{1}{x} + \frac{2x}{x^2+1} - \frac{1}{2(x+1)}\right)$$

Extra Questions

Question 4. Let $f(x) = x^3 + 2x + 4$ and g it's inverse. Without finding a formula for $g(x)$ (no seriously, don't even try) calculate $g(7)$ and then $g'(7)$.

Solution to Question 4.
$g(7) = b$ where b is the number such that $f(b) = 7$. i.e, $b^3 + 2b + 4 = 7$ and so $b = 1$ by inspection. Now, by the derivative of the inverse formula we have that

$$g'(7) = \frac{1}{f'(g(7))} = \frac{1}{f'(1)} = \frac{1}{5}.$$

Question 5. Calculate the following derivatives

(a) $y = \ln(x^2 6^x)$

(b) $y = \ln\left(\frac{x+1}{x^3+1}\right)$

(c) $y = 8^{\cos(x)}$

(d) $y = x^{e^x}$

Solution to Question 5.

(a)

$$\begin{aligned} y &= \ln(x^2 6^x) \\ y &= 2\ln(x) + x\ln(6) \\ \frac{dy}{dx} &= \frac{2}{x} + \ln(6). \end{aligned}$$

(b)

$$\begin{aligned} y &= \ln\left(\frac{x+1}{x^3+1}\right) \\ y &= \ln(x+1) - \ln(x^3+1) \\ \frac{dy}{dx} &= \frac{1}{x+1} - \frac{3x^2}{x^3+1}. \end{aligned}$$

(c)

$$\begin{aligned} y &= 8^{\cos(x)} \\ y &= e^{\ln(8)\cos(x)} \\ \frac{dy}{dx} &= -\ln(8)\sin(x)e^{\ln(8)\cos(x)} \\ &= -\ln(8)\sin(x)8^{\cos(x)}. \end{aligned}$$

(d)

$$\begin{aligned} y &= x^{e^x} \\ y &= e^{\ln(x)e^x} \\ \frac{dy}{dx} &= \frac{d}{dx}(\ln(x)e^x) \cdot e^{\ln(x)e^x} \\ &= \left(\frac{e^x}{x} + \ln(x)e^x\right) \cdot x^{e^x}. \end{aligned}$$

Question 6. Prove that $\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$. Remember that $\arcsin(x)$ is the inverse of sin after restricting the domain to $[-\pi/2, \pi/2]$.

Solution to Question 6.
Let $y = \arcsin(x)$. Then $\sin(y) = x$ and differentiating gives us that

$$\cos(y)\frac{dy}{dx} = 1.$$

Note that by Pythagoras we have that $\cos(y) = \sqrt{1-\sin^2(y)}$ since we have restricted the domain so that $\cos(y)$ is positive. Hence altogether we have

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}.$$