MATH 31B: Week 1

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Questions

Question 1. Without a calculator, calculate the following

(a) $\log_3(27)$

(b) $\ln(e^3) + \ln(e^4)$

(c) $\log_7(49^2)$

Solution to Question 1.

(a) $\log_3(27) = \log_3(3^3) = 3\log_3(3) = 3$

(b) $\ln(e^3) + \ln(e^4) = 3\ln(e) + 4\ln(e) = 7$

(c) $\log_7(49^2) = 2\log_7(49) = 2\log_7(7^2) = 4\log_7(7) = 4$

Question 2. Solve for the unknown.

1. $2^{x^2-2x} = 8$
2. $\ln(x^4) - \ln(x^2) = 2$

Solution to Question 2.

1. We have that $x^2 - 2x = \log_2(8) = 3$. Solving this quadratic gives $x = 3, -1$.
2. By log rules we get that $2\ln(x) = 2$ and so $\ln(x) = 1$ and we then get $x = e$.

Question 3. Differentiate the following

(a) $f(x) = e^{\cos(x)}$

(b) $f(x) = 2^x$, hint $2 = e^{\ln(2)}$.

(c) $f(x) = \ln(3x^3 + 2x)$

(d) $f(x) = \ln\left(\dfrac{x^2+1}{x-1}\right)$

Solution to Question 3.

(a) $f'(x) = -\sin(x)e^{\cos(x)}$

(b) We have that $f(x) = e^{\ln(2)x}$ and so by chain rule $f'(x) = \ln(2)e^{\ln(2)x} = \ln(2)2^x$.

(c) $f'(x) = \dfrac{9x^2+2}{3x^3+2x}$

(d) For this one we can use the chain rule straight away, however it is a bit simpler if we first break apart the log using the log rules.

$$\begin{aligned} f(x) &= \ln\left(\frac{x^2+1}{x-1}\right) \\ &= \ln(x^2+1) - \ln(x-1). \end{aligned}$$

Hence differentiating gives

$$f'(x) = \frac{2x}{x^2+1} - \frac{1}{x-1}.$$

Homework Questions

7.2.5, 7.1.31, 7.1.48, 7.3.22, 7.3.29, 7.3.73

Extra Questions

Question 4. For each of the following functions, determine if they have an inverse or not. If they do, find it. If they don't, restrict the domain such that they do have an inverse and then find it.

(a) $f(x) = x^3 + 3$,

(b) $f(x) = (x-3)^2$,

(c) $f(x) = \dfrac{3x+2}{5x-1}$. Note that the domain of this function is $D = \{x : x \neq \frac{1}{5}\}$.

Solution to Question 4.

(a) Differentiating gives that $f'(x) = x^2 > 0$ when $x \neq 0$, and since this is zero at only one point we see that f is strictly increasing. Hence it is one-to-one and has an inverse. Solving $y = f(x)$ for y gives the inverse as $f^{-1}(x) = \sqrt[3]{x-3}$

(b) Geometrically, this is a parabola and doesn't pass the horizontal line test and hence isn't one-to-one. Alternatively, we can also observe that $f(2) = f(4) = 1$ and so isn't one-to-one. We can restrict the domain to $[3, \infty)$ and so it is one-to-one on this domain. To find the inverse we solve for y as usual. Note that taking square roots gives two possible answers.

$$\begin{aligned} y &= (x-3)^2 \\ \pm\sqrt{y} &= x-3 \end{aligned}$$

Since we have restricted to the domain $[3, \infty)$, the left side is positive and so we take the positive root which gives us

$$\begin{aligned} \sqrt{y} &= x-3 \\ \sqrt{y}+3 &= x. \end{aligned}$$

and so we find the inverse to be $f^{-1}(x) = \sqrt{x}+3$. Note that if we had restricted the domain to $(-\infty, 3]$ instead, then we would have had to take the negative root giving $-\sqrt{y} = x-3$ as the right hand side is negative over this domain.

(c) Differentiating we find that $f'(x) = \dfrac{13}{(5x-1)^2} > 0$ for all x in it's domain. However, the domain doesn't include the point $x = \frac{1}{5}$ and so this by itself doesn't yet prove that the function f is one-to-one (why?). This question is more easily done by just finding the inverse directly (this then implies that the function is one-to-one). We rearrange as usual

$$\begin{aligned} y &= \frac{3x+2}{5x-1} \\ 5xy - y &= 3x+2 \\ 5xy - 3x &= y+2 \\ x(5y-3) &= y+2 \\ x &= \frac{y+2}{5y-3}. \end{aligned}$$

Hence the inverse is $f^{-1}(x) = \dfrac{x+2}{5x-3}$.

Question 5. What is the derivative of $f(x) = x^x$?

Solution to Question 5.
It is not $x \cdot x^{x-1}$! We must first rewrite this as $f(x) = e^{x\ln(x)}$ and then by chain and product rule we get

$$\begin{aligned} f'(x) &= e^{x\ln(x)} \frac{d}{dx}\left(x\ln(x)\right) \\ &= x^x\left(1 + \ln(x)\right) \end{aligned}$$