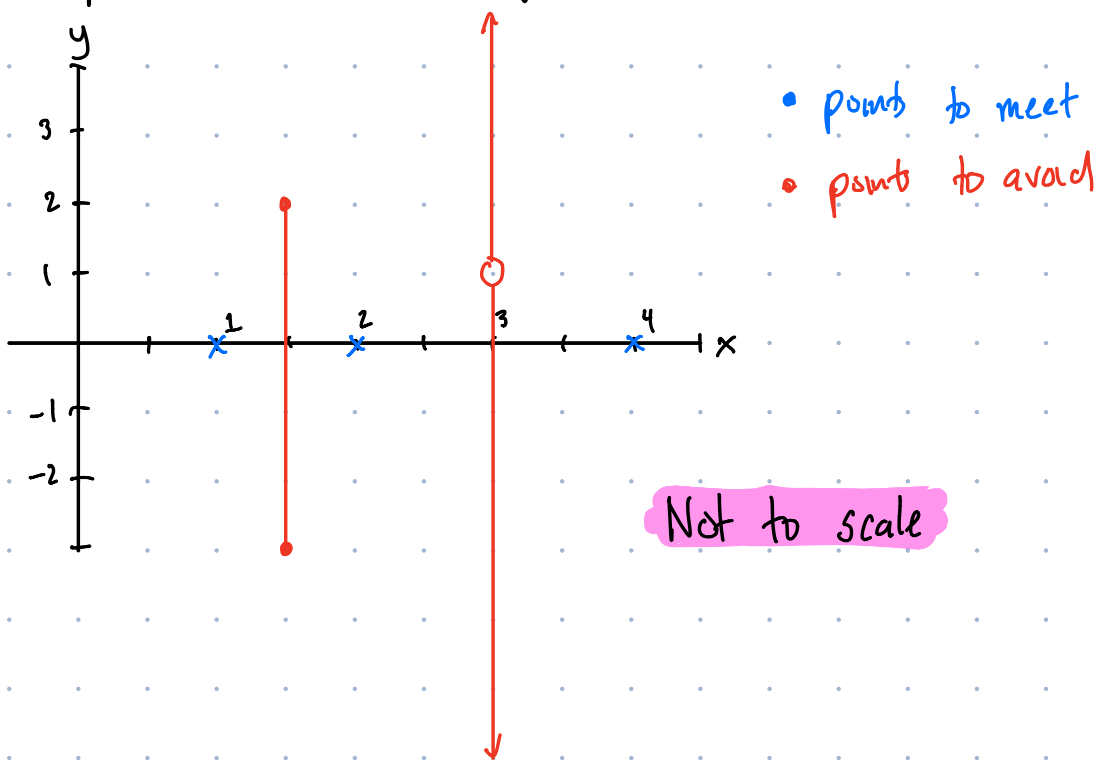
Problem 1

Problem 2

The points and line on the xy-plane looks like:



We build our function piecewise and deal with this in sections.

For $-\infty < x \leq 1$, there are no difficulties and take $f(x) = 0$ here.

For $1 \leq x \leq 2$, we need to avoid the line S_1, and connect the points P_1, P_2. We can do this in a bunch of ways. We will start by taking the polynomial that connects these points:

$$y = (x-1)(x-2).$$

This however does intersect the line S_1 as $x = 1.5 \Rightarrow y = -\frac{1}{4} \in S_1$. Hence we will scale this by a number large enough so that the peak of the polynomial avoids the line. i.e, we will pick a scaling factor of 16, so the polynomial becomes $y = 16(x-1)(x-2)$.

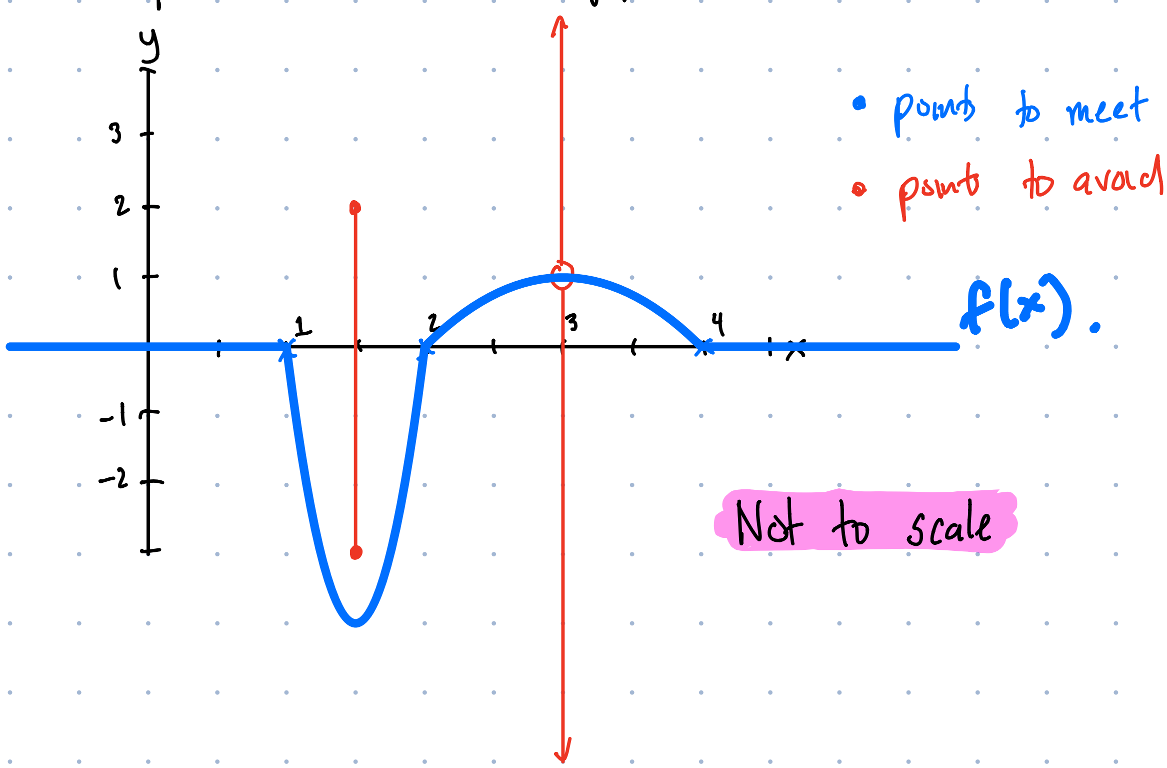
Then we have at $x = 1.5 \rightarrow y = -4$ and so this avoids the line S_1.

For $2 \leq x < \infty$, we must find an equation that connects the points P_2, P_3 and avoids the line S_2. Since the only way to avoid S_2 is by going through the point $Q = (3,1)$. We could use a similar strategy as before to fit a polynomial through these three points, or alternatively, notice that each point is 1 unit distant from the point $(3,0)$. Hence they fit on the upper semicircle $y = \sqrt{1-(x-3)^2}$, $2 \leq x \leq 3$.

Bring this all together, we have the function:

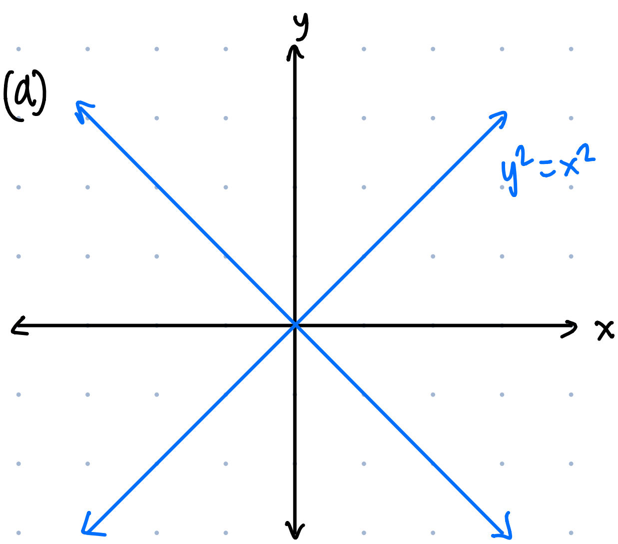
$$f(x) = \begin{cases} 0 & \text{if } -\infty < x \leq 1 \\ 16(x-1)(x-2) & \text{if } 1 < x \leq 2 \\ \sqrt{1-(x-3)^2} & \text{if } 2 < x \leq 3 \\ 0 & \text{if } 3 < x < \infty \end{cases}$$

The points and line on the xy-plane looks like:



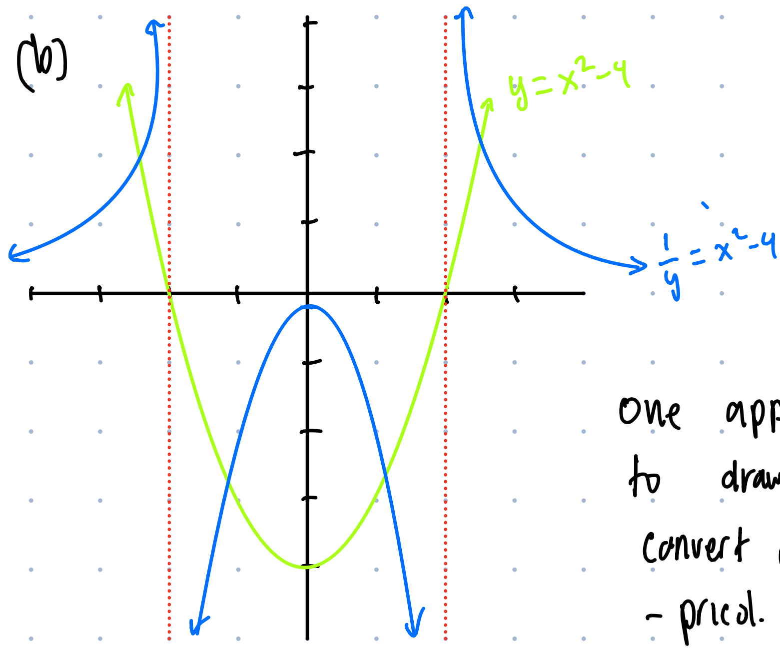
Problem 3

(a)



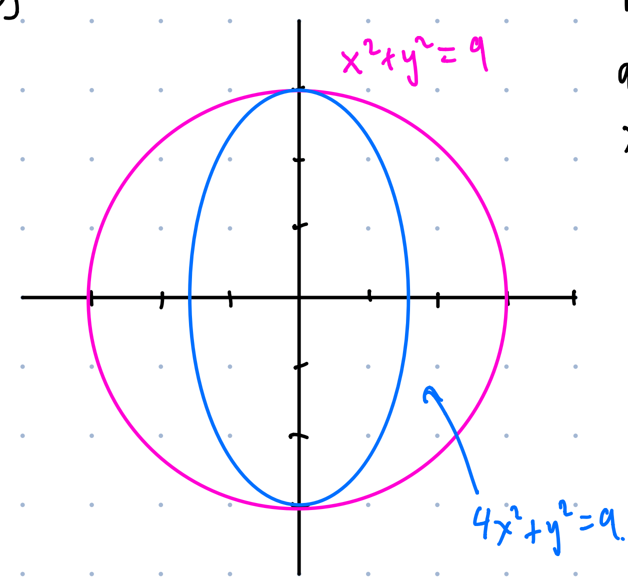
Not a function.

(b)



One approach to this question is to draw $y=x^2-4$ first and then convert each y-value to its reciprocal. This turns zeroes to asymptotes, and $a \mapsto 1/a$.

(C)



Rearranging gives $4x^2+y^2=9$. This is almost the circle $x^2+y^2=9$ but x has been swapped with 2x. This has the effect of scaling the graph in the x-direction by 1/2.

Problem 4

(a)

Given two points (x_0, y_0), (x_1, y_1), the equation that joins them is given by:

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0).$$

Hence, plug in 1, 1+a into f to get the y-coords:

$f(1) = 4$,

$f(1+a) = (1+a)^2 + (1+a) + 2 = a^2 + 2a + 1 + 1 + a + 2 = a^2 + 3a + 4.$

The the equation becomes:

(with $x_0 = 1, y_0 = 4$, $x_1 = 1+a$, $y_1 = a^2 + 3a + 4$)

$$y - 4 = \frac{a^2 + 3a + 4 - 4}{1 + a - 1}(x - 1)$$

$$y = \frac{a^2 + 3a}{a}(x - 1) + 4$$

secant line:

$$y = (a + 3)(x - 1) + 4 \qquad \text{as} \qquad a > 0$$

(b) The slope of the above line is $a+3$. Hence the slope as a function is

$$m(a) = a + 3.$$

(c) We get:

$$m(1) = 4$$
$$m(0.1) = 3.1$$
$$m(0.01) = 3.01$$

This gets closer and closer to 3.

(d) Just as above, $m(0) = 3$. Geometrically, I expect this to be the slope of the tangent line at $x = 1$.

(e) If we take $(x_0, y_0) = (x_1, y_1) = (1, 4)$ in the equation for a line we get:

$$y - 4 = \frac{4-4}{1-1}(x-1)$$

$$y - 4 = \frac{0}{0}(x-1)$$

This doesn't make sense as $\frac{0}{0}$ doesn't make sense.

(f) We got from (c) that we expect the tangent line to have slope 3. Hence the tangent line should be:

$$y - 4 = 3(x-1)$$
$$y = 3x + 1.$$