MATH 31A: Week 1

TA: Ben Szczesny Last updated: 2018/10/02

Discussion Questions

Question 1. Use the formal definition of limit to prove that $\lim_{x \to 3} 2x + 4 = 10$.

Solution to Question 1.
We first relate the gap $|f(x) - 10|$ to $|x - 3|$. We get that $|f(x) - 10| = |2x - 6| = 2|x - 3|$, and we see that the gap is twice a big as $|x - 3|$. Hence it follows that for any $\epsilon > 0$, if $|x - 3| < \epsilon/2$, then we have that $|f(x) - 10| = 2|x - 3| < 2 \times \epsilon/2 = \epsilon$. Hence we have proven that $\lim_{x \to 3} 2x + 4 = 10$.

Question 2. In some cases, numerical investigations of limits can be misleading. Consider the function $f(x) = \cos\left(\frac{\pi}{x}\right)$.

(a) Evaluate $f(x)$ at $x = \pm\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots,$.

(b) Does $\lim_{x \to 0} f(x)$ exist?

Solution to Question 2.

1. Substituting in $x = \frac{1}{2n}$ gives us $f(1/2n) = 1$.
2. Despite the fact that the previous part suggests this function has a limit of 1 (the points are getting closer and closer to zero and they evaluate to 1), this function does not have a limit at $x = 0$. We can see this by taking a different sequence of points $x = \frac{1}{3}, \frac{1}{5}, \cdots, \frac{1}{2n+1}, \cdots$ and evaluating f at these points gives us $f\left(\frac{1}{2n+1}\right) = -1$. Hence we see that approaching zero a different way gives us a different number and so we see that the limit does not exist.

Question 3. Evaluate the following limits assuming that $\lim_{x \to 2} f(x) = 2$ and $\lim_{x \to 2} g(x) = 5$.

(a) $\lim_{x \to 2} f(x)g(x)$

(b) $\lim_{x \to 2} \frac{g(x)}{f(x)}$

(c) $\lim_{x \to 2} 2g(x) - 3$

(d) $\lim_{x \to 2} \frac{4f(x)}{g(x) - 2}$

Solution to Question 3.
Note: you should state which limit laws you are using for each step of your solution in the following questions

(a) 10

(b) 5/2

(c) 7

(d) 8/3

Question 4. Give an example where $\lim_{x \to 0}(f(x) + g(x))$ exists but neither $\lim_{x \to 0} f(x)$ nor $\lim_{x \to 0} g(x)$ exists.

Solution to Question 4.
Consider $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$. Then neither $\lim_{x \to 0} f(x)$ nor $\lim_{x \to 0} g(x)$ exist. Yet,

$$\lim_{x \to 0}(f(x) + g(x)) = \lim_{x \to 0} 0 = 0.$$

Question 5. Determine the points of discontinuity (removable, jump, infinite or none of these) and whether the function is left- or right-continuous.

(a) $f(x) = \left\lfloor \frac{x}{2} \right\rfloor$

(b) $f(x) = \dfrac{x+1}{4x-2}$

(c) $f(x) = \begin{cases} x^2 & \text{for } x \leq 1 \\ 2 - x & \text{for } x > 1 \end{cases}$

Solution to Question 5.

(a) We have that $f(x) = n$ for $x \in [2n, 2n+2)$. Hence we see that f has a jump discontinuity at each point $2n$, $n \in \mathbb{Z}$. It is also right continuous at these points.

(b) The is a quotient of two continuous functions and so it is continuous for all points where the denominator is not zero. Hence we have that $x = 1/2$ is a point of discontinuity (it is not defined here). This is an infinite discontinuity.

(c) Since $1 = f(1) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$ we see that f is a continuous function.

Question 6. Suppose that $\lim_{t \to 3} tg(t) = 12$. Show that $\lim_{t \to 3} g(t)$ exists and is equal to 4.

Solution to Question 6.
We have by limit laws that $\lim_{t \to 3} \frac{1}{t} = \frac{1}{3}$. Hence by limit laws we get that

$$\begin{aligned} \lim_{t \to 3} g(t) &= \lim_{t \to 3} \frac{1}{t} \cdot tg(t) \\ &= \lim_{t \to 3} \frac{1}{t} \cdot \lim_{t \to 3} tg(t) \\ &= \frac{1}{3} \cdot 12 \\ &= 4. \end{aligned}$$

Note, it is *incorrect* to say that

$$12 = \lim_{t \to 3} tg(t) = \lim_{t \to 3} tg(t) = \left(\lim_{t \to 3} t\right) \cdot \left(\lim_{t \to 3} g(t)\right) = 3 \lim_{t \to 3} g(t).$$

Hence $\lim_{t \to 3} g(t) = 4$. This is incorrect since because we do not know that $\lim_{t \to 3} g(t)$ exists, we can not use the limit law to say that $\lim_{t \to 3} tg(t) = (\lim_{t \to 3} t) \cdot (\lim_{t \to 3} g(t))$.