Q1: Find one generator of the ideal in $\mathbb{Z}[i]$ generated by 2 and $5+3i$.

we have that $2 = (1-i)(1+i)$
and $5+3i = 3(1+i) + 2$

$$= 3(1+i) + (1-i)(1+i)$$

$$= (3+(1-i))(1+i)$$

Hence $(2, 5+3i) \subseteq (1+i)$ and as
$1+i = (5+3i) - 2(2+i)$ we have
$1+i \in (2, 5+3i)$ and

$$(2, 5+3i) = (1+i)$$

So $1+i$ is such a generator. $\blacksquare$

Q2: Prove $x^7 + y^7 + z^7$ is irreducible in $\mathbb{C}[x,y,z]$.

View $x^7+y^7+z^7$ as a polynomial in $R[z]$ where $R = \mathbb{C}[x,y]$.
Now, $x^7+y^7 = (x+y)(x^6 - x^5y + \cdots + y^6)$
where $x+y$ is prime in R.

Moreover, $x+y$ doesn't divide $x^6 - x^5y + \cdots + y^6$.

One can see this by looking at the image of $x^6 - x^5y + \cdots + y^6$ in $R/(x+y)$ which is equal to $7\bar{y}^6 \neq 0$ since $\mathbb{C}$ has characteristic 0.

Now, $x+y \mid x^7+y^7$ and $(x+y)^2 \nmid x^7+y^7$.
Hence by eisenstein's gives us $x^7+y^7+z^7$ irreducible

Q3: Let $R = \mathbb{Z}[x,y]$ be a polynomial ring and let I be the ideal of R generated by x and y. Construct an exact sequence of R-modules

$$0 \to R \to R \oplus R \to I \to 0.$$

Let $f: R\oplus R \to I$ by $(f_1, f_1) \mapsto f_1 x + f_2 y$. This is easily checked to be an R-module hom.

Now, $(f_1, f_2) \in \ker f \Leftrightarrow f_1 x + f_2 y = 0$

It follows that $f_1 = f_1' y$ and $f_2 = f_2' x$ for some $f_1', f_2' \in R$ and $(f_1' + f_2')xy = 0$

$\Rightarrow f_1' + f_2' = 0$ since R ID.

Hence $\ker f = \{(gy, -gx) \in R\oplus R \mid g \in R\}$.

and so take $h: R \to R\oplus R$ given by $g \mapsto (gy, -gx)$ which is easily checked to be an R-module hom.

Hence:

$$0 \to R \xrightarrow{h} R\oplus R \xrightarrow{f} I \to 0$$

is exact.

Q4: Let $a_1, a_2, \dots, a_n$ be elements of a commutative ring R generating the unit ideal in R. Show that the submodule M in R^n consisting of all n-tuples $(x_1, x_2, \dots, x_n) \in R^n$ such that $a_1 x_1 + a_2 x_n + \cdots + a_n x_n = 0$ is projective.

We have the exact sequence

$$0 \to M \xrightarrow{i} R^n \xrightarrow{f} R \to 0$$

where $f(x_1, \dots, x_n) = \sum a_i x_i$ and i injection. Since $(a_1, a_2, \dots, a_n) = 1$, f is surjective and it is exact at M by definition

Now, $\sum a_i b_i = 1$ for some b_i.

Hence let $g(r) = (b_1 r, \dots, b_n r)$ (easily checke
R-mod h

This is such that $fg = id_R$

Hence the sequence splits and

$R^n = M \oplus R$. Hence M is projective

Q5 Let M, N, and P be finitely generated modules over a PID R such that $M \oplus P \cong N \oplus P$. Are the modules M and N necessarily isomorphic? Explain.

Yes. For finitely-generated modules over a PID, two modules are isomorphic if and only if they share the same set of elementary divisors and free rank.

So $M \oplus P$, $N \oplus P$ have same free rank and elementary divisors. Since these behave well with direct sum, M and N must have same free rank and elementary divisors.

Hence $M \cong N$. ∎