Week 8: Fields

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Question 1. Show that the order of a finite field $\mathbb{F}$ cannot be divisible by two distinct primes.

Question 2. Show for a field $\mathbb{F}$ the following are equivalent.

1. There exists $\alpha_1, \cdots, \alpha_n \in \mathbb{F}$ such that the only subfield of $\mathbb{F}$ containing all the α_i 's is $\mathbb{F}$ itself.
2. $\mathbb{F}$ is the fraction field of $\mathbb{Z}[x_1, \cdots, x_n]/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$.

Question 3. Let $a/b \in \mathbb{Q}$. Show that $\cos(\frac{a}{b}\pi)$ is algebraic over $\mathbb{Q}$.

Question 4. Given a commutative ring B and subring A, we say that an element of $x \in B$ is integral over A if there exists a monic polynomial $f(t) \in A[t]$ such that $f(x) = 0$. Note that if A and B are fields, then this is the same definition for algebraic over. Show that the following are equivalent:

1. x is integral over A ;
2. The subring $A[x]$ of B generated by A and x is a finitely generated A-module;
3. There exists a subring C of B containing $A[x]$ and which is finitely generated as an A-module;
4. There exists a finitely generated A-submodule M of B such that $xM \subseteq M$ and M is faithful over $A[x]$ (That is the map given by the action $A[x] \to Hom(M)$ is injective).

Question 5. Show that any automorphism on the real numbers is identity.