Week 7: Misc Module stuff

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Question 1. Let R be a unital associative ring. Show that the left ideals of $M_n(R)$ are in one to one correspondence with left R-submodules of R^n.
As an extra question, what do the two sided ideals of $M_n(R)$ look like?

Question 2. (Spring '18) View $\mathbb{Z}^n$ as column vectors. Show that for any left ideal $I \subseteq M_n(\mathbb{Z})$ the subgroup $I\mathbb{Z}^n \subseteq \mathbb{Z}^n$ has finite index.

Question 3. (Fall '16) Let A be an integral domain with field of fractions F. Suppose $\mathfrak{a}$ is an ideal of A. Show that $\mathfrak{a}$ is a finitely generated projective A-module if there exists a A-submodule $\mathfrak{b}$ of F such that $\mathfrak{a}\mathfrak{b} = A$ in F.

Question 4. (Spring '19) Let R be a commutative local ring and M a finitely generated projective module. Show that M is R-free.

Question 5. (Fall '17) Let R be a commutative Noetherian ring and A a finitely generated R-algebra (not necessarily commutative). Let B be an R-subalgebra of the center $Z(A)$. Assume A is a finitely generated B-module. Show that B is a finitely generated R-algebra.

Question 6. (Spring '18) Let B be a commutative Noetherian ring, and let A be a Noetherian subring of B. Let I be the nilradical of B. If B/I is finitely generated as an A-module, show that B is finitely generated as an A-module.

HINTS BELOW. FOLD HERE.

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1. Think about elementary row operations as the left action of certain matrices.
2. Entries in a column must be multiples of a common element. Think about gcds.
3. We must have that $\sum a_i b_i = 1$. Try and use this to first show finitely generated and then use this to construct a split exact sequence.
4. Local rings suggest Nakayama and finitely projective suggests a certain split exact sequence.
5. This is essentially the Artin-Tate lemma. The main idea of this proof is that if you can find an intermediate finitely generated R-algebra $B_0 \subseteq B$ such that A is a noetherian B_0-module, then you can view B as a B_0-submodule.
6. I couldn't figure out a proof of this in time and it's annoying me. Let me know if you come up with one!