Week 6: Homology and stuff.

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Question 1. Let Γ be a directed finite graph with vertex set $V = \{v_1, \cdots, v_n\}$ and edge set $E = \{e_1, \cdots, e_m\}$. The incident matrix M is a $n \times m$ matrix with (i, j)-th entry $+1$ if edge e_j starts at vertex v_i and -1 if it ends there. All other entries are zero. Let C_0 be the free R-module generated by the vertices and C_1 the free R-module generated by the edges. Let $C_n = 0$ for $n \neq 0, 1$ and $d_n = 0$ if $n \neq 1$ and $d_1 = M$. This forms a chain complex. Show that if Γ is connected then $H_0(C)$ and $H_1(C)$ are free of rank 1 and $m - n + 1$ respectively.

Question 2. Use the Snake lemma to show that a short exact sequence

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

gives a long exact sequence in homology

$$\cdots \to H_i(A_\bullet) \to H_i(B_\bullet) \to H_i(C_\bullet) \to H_{i-1}(A_\bullet) \to \cdots$$

Question 3. Show that $\mathrm{Tor}_0^R(M, N) = M \otimes_R N$.

Question 4. Guess why a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

gives a long exact sequence in Tor

$$\cdots \to \mathrm{Tor}_2^R(M'', N) \to \mathrm{Tor}_1^R(M', N) \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^R(M'', N) \to M' \otimes_R N \to \cdots$$

Question 5. Show that if R is a PID, then $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 1$.

Question 6. Show that if $r \in R$ is not a zero divisor and N is an R-module. Then $\mathrm{Tor}_1^R(R/(r), N)$ is isomorphic to the r-torsion of N.