Week 4

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The following is sometimes called Nakayama's lemma. I prefer to call it gerneralized Cayley-Hamilton.

Theorem 1. *Let R be a commutative ring and I and ideal. Let M be a finitely generates R-module and f an endormorphism of M. If $f(M) \subseteq IM$ then there exists an n and $a_i \in I$ such that*

$$f^n + a_{n-1}f^{n-1} + \cdots a_1 f + a_0 = 0 \text{ in } End(M)$$

Question 1. The following are all called Nakayama's lemma. Prove all of them. Let R be a commutative ring and M a finitely generated R-module.

1. Let I be an ideal of R such that $IM = M$. Then there exists an $x \in R$ such that $x = 1$ in R/I and $xM = 0$. An equivalent formulation to this which is easier to remember is that if $IM = M$ then there exists an element $i \in I$ such that $im = m$ for all $m \in M$.
2. Let I be an ideal contained in $Rad(R)$. Then $IM = M$ implies that $M = 0$.
3. Let N be submodule of M and $I \subseteq Rad(R)$. Then if $M = IM + N$ then $M = N$.
4. Suppose that R is a local ring with maximal ideal $\mathfrak{m}$ and residue field $F = R/\mathfrak{m}$. Suppose we have elements x_i of M such that their images in $M/\mathfrak{m}M$ form an F-basis. Then the x_i generate M.

Question 2. Suppose R is an integral domain which isn't a field and let $F = R_{(0)}$. Show that F cannot be a finitely generated R-module.

Question 3. Let M be a finitely generated R-module for commutative ring R. Show that every surjective endomorphism is an isomorphism.

Question 4. Let S be a multiplicative set of ring R. Show that the functor $S^{-1} : mod(R) \to mod(S^{-1}R)$ is exact. That is, maps short exact sequences to short exact sequences.

Definition 1. We call a property P of R-modules local if a module M has property P if and only if $M_\mathfrak{q}$ has property P for all prime ideals $\mathfrak{q}$.

Question 5. Show the following are local properties:

1. A module being trivial.
2. A R-homomorphism being injective.
3. A module being torsion free.

Question 6. Prove the following version of Nakayama's Lemma: