Math210B: Week 2

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Factorization in polynomial rings

Throughout this work sheet let R be a UFD. Here are some definitions:

Definition 1. let $\{a_i\}$ be elements of R. A GCD for $\{a_i\}$ is an element $d|a_i$ for all i such that if $c|a_i$ for all i then $c|d$.

Definition 2. let $f \in R[X]$, the content of f, denoted by $cont(f)$, is the GCD of the coeffeicients of f.

Definition 3. A polynomial $f \in R[X]$ is primitive if $cont(f)$ is a unit. We can always write a polynomial as $f = cont(f)f'$ where f' is primitive. We call this the primitive part and denote it by $pp(f) = f'$.

Note, content is sometimes defined differently. Also, there is a generalisation to the following two Gauss' lemma which can be found on the Gauss lemma wiki page.

Question 1. Prove Gauss' lemma (primitivity statement): Let $f, g \in R[X]$ be primitive. Then fg is also primitive.

Definition 4. Given a domain R we let F be the ring of fractions which is

$$\{(a, b) \in R^2 \mid b \neq 0\}/\sim \text{ where } (a, b) \sim (c, d) \text{ if } ad = bc$$

we define operations $(a, b) + (c, d) = (ad + bc, bd)$ and $(a, b) \cdot (c, d) = (ac, bd)$. This makes F a field. We denote the class (a, b) by a/b.

Question 2. If you haven't seen ring of fractions before then you should do the following.

1. Show the operations of addition and multiplication are well defined. What are the identities?
2. Show that the map $a \mapsto a/1$ is injective.

Question 3. Prove Gauss' lemma (irreducubility statement): A nonconstant polynomial $f \in R[X]$ is irreducible if and only if it is irreducible in $F[X]$ and primitive in $R[X]$.

Question 4. Show that if R is a UFD then $R[X]$ is a UFD.

Question 5. Are UFD's noetherian?

Question 6. Eisenstein's Criterion: Let R be a UFD with ring of fractions F. If $f = a_0 + \cdots + a_n X^n \in R[X]$ and p is a prime element such that

1. $p \nmid a_n$;
2. $p \mid a_i$ for $i \neq n$;
3. $p^2 \nmid a_0$.

Then f is irreducible in $F[X]$.