Week 10

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Definition 1. Given a finite extension E/F, we can view an element $x \in E$ as a F-linear transformation on E via multiplication. The determinant of this transformation is called the norm and we denote it by $N_{E/F}(x)$. The trace is called the trace and denoted $tr_{E/F}(x)$.

Question 1. Given a simple algebraic extension $F(\alpha)/F$ with α having minimal polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ over F. Prove that

$$tr_{F(\alpha)/F}(\alpha) = -a_{n-1} \quad \text{and} \quad N_{F(\alpha)/F}(\alpha) = (-1)^n a_0.$$

Question 2. Let E/F be a finite extension and $\alpha \in E$. Assume that $[E : F(\alpha)] = r$. Prove that

$$tr_{E/F}(\alpha) = r tr_{F(\alpha)/F}(\alpha) \quad \text{and} \quad N_{E/F}(\alpha) = N_{F(\alpha)/F}(\alpha)^r.$$

Question 3. Suppose E/F is a finite Galois extension. Show that for $\alpha \in E$ we have that

$$N_{E/F}(\alpha) = \prod_{\sigma \in Gal(E/F)} \sigma(\alpha), \quad tr_{E/F}(\alpha) = \sum_{\sigma \in Gal(E/F)} \sigma(\alpha).$$

At this point it is a good idea to read Keith Conrad's notes "Linear independence of characters" for a good introduction to Hilbert's 90 thoerem and basic kummer theory where the trace and norm find applications. Originally I wanted to go through this but realised it would take too long.

Question 4. Suppose we have an irreducible polynomial $f \in \mathbb{Q}[X]$ of degree p, where p is prime. Assume that f has $p-2$ real roots and 2 nonreal, complex roots. What is that Galois group of f?

Question 5. (Fall '16) Let $f \in F[X]$ be an irreducible polynomial of prime degree over a field F, and let K/F be the splitting field of f. Prove there is an element in the Galois group of K/F permuting cyclically all the roots of f in K.

Question 6. (Spring '18) let $\alpha \in \mathbb{C}$ and suppose that $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is finite and coprime to $n!$ for some integer $n > 0$. Show that $\mathbb{Q}(\alpha^n) = \mathbb{Q}(\alpha)$.