MATH210B: Week 1

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Let R be a unital commutative ring. There is a topological space that we can view the ring R as a natural set of functions on. We will construct it in the first question, but first some definitions:

Definition 1. A proper ideal $I \subset R$ is prime if whenever $ab \in I$ then either $a \in I$ or $b \in I$.

Definition 2. The spectrum of a ring R, denoted by $Spec(R)$, is the set of all prime ideals.

Definition 3. The radical of an ideal $\mathfrak{a}$ is given by $r(\mathfrak{a}) = \{r \in R \mid r^n \in \mathfrak{a} \text{ for some } n\}$.

Question 1. Let R be a ring. For any subset $S \subseteq R$ let

$$V(S) = \{x \in Spec(R) \mid S \subseteq x\}.$$

These are sometimes called the vanishing sets. Show the following:

(a) Let $E \subseteq R$ and $\mathfrak{a}$ the ideal generated by E. Show that $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$

(b) $V(0) = Spec(R)$ and $V(1) = \emptyset$.

(c) For an arbitrary collection of subset E_i of R we have that

$$V(\cup E_i) = \cap V(E_i)$$

(d) For two ideals $\mathfrak{a}$, $\mathfrak{b}$ we have $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$

Observe that these show that the vanishing sets form the closed sets of a topology on $Spec(R)$ called the Zariski topology.
We can view elements $f \in R$ as functions on $Spec(R)$ by considering it's value on $P \in Spec(R)$ as $f(P) = f \pmod{P}$. So the Vanishing sets are exactly the points it is zero. Unlike normal functions, a function being everywhere zero is not enough for it to be zero as the next exercise shows us.

Question 2. Show that $f \in R$ is everywhere zero on $Spec(R)$ if and only if f is nilpotent. *Hint:* Consider the multiplicative set S which constains 1 and all powers of some element f. One can use zorn's to show that the set of all ideals not intersecting S must have a maximal and one can then show this is prime.

Question 3. Given a ring R and an ideal $\mathfrak{a}$. How are the spectrums $Spec(R)$ and $Spec(R/\mathfrak{a})$ related? What about $Spec(R)$ and $Spec(R/N)$ where N is the nilradical(set of all nilpotents)?

Question 4. Question 2 can be generalized into what's sometimes called the nullstellensatz for general rings. For a subset $S \subseteq Spec(R)$ let

$$I(S) = \cap_{P \in S} P$$

i.e, all the elements that vanish on S. Show that

$$VIV(J) = V(r(J))$$

for any ideal J. Moreover, show that V and I form a bijective correspondence between closed subsets and radical ideals.

Question 5. Professor Merkurjev asked me to explain the following, which I feel is a nice little exercise:

Show that a ring R is a product of n rings if and only if there exists n central orthogonal idempotents that sum to 1. (that is central elements e_i such that $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$).