Question 1

Let n be the order of 1 in $\mathbb{F}$. (abelian additive structure). n must be prime since if $n = km$, then $0 = n \cdot 1 = (km) \cdot 1 = (k \cdot 1)(m \cdot 1)$ and so we have zero divisors which can't happen as $\mathbb{F}$ a field.
n is called the characteristic of $\mathbb{F}$, usually denoted $\chi(\mathbb{F})$.

Now, suppose $p = \chi(\mathbb{F})$ and there exists another distinct prime q s.t $q \mid |\mathbb{F}|$

Then by cauchy's lemma, there exists a $x \in \mathbb{F}$ $x \neq 0$ s.t $q \cdot x = 0$. We also have that $p \cdot x = (p \cdot 1) \cdot x = 0x = 0$. By Bezout's lemma, as p, q distinct primes, there exists integers $a, b \in \mathbb{Z}$ s.t $ap + bq = 1$. Then $x = 1 \cdot x = (ap) \cdot x + (bq) \cdot x = 0$ a contradiction.

Question 2

① ⇒ ②

We have a ring map $\mathbb{Z}[x_1, .., x_n] \xrightarrow{\varphi} \mathbb{F}$ given by mapping $x_i \mapsto \alpha_i$

This factors to an injective map by 1st isomorphism:

$$\bar{\varphi} : \mathbb{Z}[x_1, .., x_n]/\ker\varphi \longrightarrow \mathbb{F}.$$

Since this is an injection, $\mathbb{Z}[x_1, .., x_n]/\ker\varphi$ is an ID and so $\ker\varphi$ is prime.

The universal property of localisation of rings gives

$$\begin{tikzcd}
K(\mathbb{Z}[x_1, .., x_n]/\ker\varphi) \arrow[r, "\Phi"] & \mathbb{F} \\
\mathbb{Z}[x_1, .., x_n]/\ker\varphi] \arrow[u, "q"] \arrow[ur, "\bar{\varphi}"'] \arrow[u, phantom, shift left=6ex, "\circlearrowright"] &
\end{tikzcd}$$

Note, q is an injection since it is an integral domain and so Φ is injective. Moreover, as the α_i generate $\mathbb{F}$, it is surjective.

② $\Rightarrow$ ①

Suppose we have an isomorphism:

$$\Phi \quad K(\mathbb{Z}[x_1, \dots, x_n]/p) \longrightarrow \mathbb{F}.$$

Let $A = \mathbb{Z}[x_1, \dots, x_n]/p$ and $\bar{x}_i$ the image of x_i in A.

I claim that $\bar{x}_1/1, \dots, \bar{x}_n/1$ generate $K(A)$ as a field

Question 3

Let $z = e^{a/b \, i\pi} = \cos(\frac{a}{b}\pi) + i\sin(\frac{a}{b}\pi)$.

then $z^b = \pm 1$ and so z algebraic over $\mathbb{Q}$.

Then $\cos(\frac{a}{b}\pi) = \frac{z + z^{-1}}{2}$ and so algebraic over $\mathbb{Q}$.

Note, in general, if α algebraic over field $\mathbb{F}$, then every element in $\mathbb{F}(\alpha)$ algebraic over $\mathbb{F}$ since $|\mathbb{F}(\alpha) : \mathbb{F}|$ finite.

Question 4

(1) $\Rightarrow$ (2) clear

(2) $\Rightarrow$ (3) take $C = A[x]$

(3) $\Rightarrow$ (4) Take $M = C$. Since $A[x] \subseteq C$, this is enough to ensure that $xC \subseteq C$ by looking at the generators of C. (as A-module)

Moreover since $A[x] \subseteq C$, this ensures M faithful over $A[x]$, since $A[x]$ is faithful over itself

(4) $\Rightarrow$ (1) By generalised-cayley hamilton, we have

that there exists

$$x^n + a_{n-1} x^{n-1} + \cdots + a_n = 0 \quad \text{in} \quad \mathrm{Hom}(M)$$

Since faithful, we have $x^n + a_{n-1} x^{n-1} + \cdots + a_n = 0$ in $A[x]$.

Question 5.

Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is an automorphism.
If $x \in \mathbb{R}$ is positive, then $\sqrt{x}$ positive
and $\phi(x) = \phi(\sqrt{x})^2 > 0$. Hence if $x > 0 \Rightarrow \phi(x) > 0$.

Now, if $x < y$, then $\phi(y - x) > 0 \Rightarrow \phi(x) < \phi(y)$.

If $x = a/b$ for some $a, b \in \mathbb{Z}$, then $\phi(x) = \phi(a)/\phi(b)$

$= a/b$ since ϕ is fixed on $\mathbb{Z}$ as $\phi(1) = 1$ by defn.

Hence $\phi(a/b) = a/b$.

Now. Let $x \in \mathbb{R}$. suppose $x < \phi(x)$. Then
there exists a rational q, s.t $x < q < \phi(x)$
and as $x < q$, $\phi(x) < \phi(q) = q$. This gives a
contradiction. Hence $\phi(x) \geq x$. We get a
similar contradiction from $\phi(x) < x$ and so
we conclude $\phi(x) = x$.