Solutions for week 7

Q1.

I'm going to look at right ideals instead as I don't want to deal with row vectors. The idea is that every right ideal of $M_n(R)$ is of the form $[P\,P \ldots P]$ where $P \leq R^n$ is a right submodule (take R^n to be column vectors)

Now, suppose $I \leq M_n(R)$ is a right ideal. By acting on the right by diagonal matrices, we see that each column is a submodule of R^n. By acting via permut. matrices, we see that the columns must be the same submodule.
Hence, Let $C(I)$ be the submodule given by a column. Let $S(P) = [P \ldots P]$ for some $P \leq R^n$ submodule. It is clear we have $CS = \mathrm{id}_{\{\text{submodules } R^n\}}$ and $SC = \mathrm{id}_{\text{ideals of } M_n(R)}$. Hence these are inverses.

The left ideal situation is the same except we take rows instead.

The two sided ideals must then be the same twosided ideal of R in each entry.

Q2 Assume I nonzero.

Any $\mathbb{Z}$-submodule of $\mathbb{Z}^n$ is free of rank $k \leq n$

in particular, they are in the form $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k$

where $v_i \in \mathbb{Z}^n$ (column vectors).

From the previous question, we then conclude that any left ideal $I \subseteq M_n(\mathbb{Z})$ is of the form

$$I = \sum_{i=1}^{k} \left(\mathbb{Z}v_i \ \mathbb{Z}v_i \cdots \mathbb{Z}v_i \right)^T$$

Let v_{ij} be the jth component of v_i

$$\text{Then } I\mathbb{Z}^n = \sum_{i=1}^{k} \left(\mathbb{Z}v_i \cdots \mathbb{Z}v_i \right)^T \mathbb{Z}^n$$

$$= \sum_{i=1}^{k} \begin{pmatrix} \mathbb{Z}\gcd_j(v_{ij}) \\ \vdots \\ \mathbb{Z}\gcd_j(v_{ij}) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{Z}\gcd_{ij}(v_{ij}) \\ \vdots \\ \mathbb{Z}\gcd_{ij}(v_{ij}) \end{pmatrix}$$

$$= \gcd_{ij}(v_{ij})\mathbb{Z}^n.$$

Since I nonzero, $\gcd_{ij}(v_{ij}) \neq 0$ and so

$\mathbb{Z}^n / \gcd_{ij}(v_{ij})\mathbb{Z}^n \cong \mathbb{Z}/\gcd_{ij}(v_{ij})\mathbb{Z}$ is finite.

Q3

we show $\underline{a}$ is finitely generated first
Since $\underline{a}\underline{b} = A$, there exist $a_i \in \underline{a}$, $b_i \in \underline{b}$
s.t $\sum_{i=1}^{n} a_i b_i = 1$

Then for all $a \in \underline{a}$, we have
$a = a(\sum a_i b_i) = \sum a b_i a_i$ and as $\underline{a}\underline{b} = A$,
$ab_i \in \underline{a}$ for all i. Hence $\{a_i\}$ gives a generating set for $\underline{a}$.

To show $\underline{a}$ is projective, we show its a direct summand of A^n. Since f.g we have SES of A-mods:

$$0 \to \ker f \to A^n \xrightarrow{f} \underline{a} \to 0$$

define a A-hom $g: \underline{a} \to A^n$ by $g(a) = \sum a b_i e_i$
where e_i is the idempotent that corresponds to the ith factor of A^n.

we then have $fg(a) = \sum a b_i a_i = a$ and so the sequence splits and we are done. ∎

Q4

let $x_1, \dots, x_n$ be a minimal set of generators for M.

There exists a SES of R-modules

$$0 \to N \to R^n \to M \to 0$$

Since M projective, $R^n = M \oplus N$.

Apply the functor $- \otimes R/I$, we get

$$(R/m)^n = (M/mM) \oplus (N/mN)$$

R/m is a feild, and any basis of M/mM pulls back to a generating set of M by nakayama.

Hence by minimality and dimension, $N/mN = 0$.

therefore, we conclude by Nakayama $N = 0$ and so $M = R^n$.

Q5.

This is essentially the Artin-Tate Lemma.

Let $\{x_1, \ldots, x_m\}$ be the generators of A as an R-algebra and $\{y_1, \ldots, y_n\}$ the generators of A as a B-module

$$\left.\begin{aligned} &\text{Then} \quad x_i = \sum_{j=1}^{n} b_{ij} y_j \quad \text{for some } b_{ij} \in B \\ &\text{and} \quad y_i y_j = \sum_{k=1}^{n} b_{ijk} y_k \quad \text{for some } b_{ijk} \in B \end{aligned}\right\} *$$

Let B_0 be the R-subalgebra of A generated by $\{b_{ij}, b_{ijk}\}$. So B_0 is a f.g R-algebra and by Hilbert basis, B_0 is noetherian as a ring. (Note, $B_0 \subseteq B \subseteq Z(A)$ and so everything is commutative here)

Observe that via the relations, $(*)$, A is generated as a B_0-module with generators y_i, and so by Hilbert basis, A is a noetherian B_0-module. Since B is a B_0-submodule of A, it is finitely generated as a B_0-submodule and as B_0 is a f.g R-algebra, we conclude B is a f.g R-algebra.

∎

Question 6

Solution 1 (by Ben Spitz)

Let $\{x_1, \ldots, x_m\}$ be a finite set of elements of B that generate B/I over A. Since B noetherian, I is f.g as a B-module. Let $\{i_1, \ldots, i_n\}$ be generators of I as a B-module.

Then $B = I + \sum_{i=1}^{m} A x_i$, so enough to show

I f.g over A.

$$I = \sum_{j=1}^{n} B i_j = \sum_{j=1}^{n} \left(I + \sum_{i=1}^{m} A x_i \right) i_j$$

$$= \sum_{j=1}^{n} I i_j + \sum_{j=1}^{n} \sum_{i=1}^{m} A x_i i_j$$

So enough to show each $I i_j$ f.g over A.

we repeat this argument and it's then enough to show $I i_j i_k$ for all j, k is f.g over A.

Then enough to show $I i_j i_k i_l$ f.g over A

etc... eventually this must be zero since I is nilpotent and so by induction we are done ∎

solution 2

After seeing Ben's solution, I realised the following observation is what I was missing to make my argument work:

claim: If I have two rings $A \subseteq C$ s.t C is f.g as an A-module, then all f.g C-modules M are also f.g A-modules.

Proof: we have $C = \sum_{i=1}^{n} A a_i$ for some a_i.

If M f.g C-module with generators $x_1, .., x_m$.

then $M = \sum_{i=1}^{m} C x_i = \sum_{i=1}^{m} \sum_{j=1}^{n} A a_j x_i$. Hence $a_j x_i$

generate M as an A-module.

Now, since B noeth. I is f.g and as each element nilpotent, we conclude $I^N = 0$ for some N. we then have a filtration of A-modules:

$$0 = I^N \subseteq I^{N-1} \subseteq \dots \subseteq I^1 \subseteq I^0 = B.$$

As B noeth., each I^k is f.g B-module and so I^k/I^{k+1} are f.g B/I-modules. By the above claim, we conclude I^k/I^{k+1} are f.g A-modules. Since the filtration is of finite length with f.g quotients

We conclude B is a f.g A-module.