Week 6

Chain complexes of R-modules

A chain complex $C_\bullet$ of R-modules is a family $\{C_n\}_{n\in\mathbb{Z}}$ of R-modules, together with maps $d = d_n : C_n \to C_{n-1}$ such that $d \circ d = 0$.

- d_n are called differentials
- $Z_n = \ker d_n$ the n-cycles
- $B_n = \operatorname{Im} d_{n+1}$ the n-boundaries
- $H_n(C_\bullet) = Z_n / B_n$ the n-th homology module of $C_\bullet$

- A morphism of chain complexes $f : C_\bullet \to D_\bullet$ is a family of maps $\{f_i\}_{i\in\mathbb{N}}$ such that $f_i : C_i \to D_i$ and $df_i = f_{i+1} d$.

- This forms the category of R-module complexes $Ch(\text{mod}_-R)$.

Question 1:

C_0 has basis $(V_1, V_2 - V_\phi, V_3 - V_\phi, \dots, V_n - V_\phi)$.

Now, since T connected, the exists a path from V_i to V_ϕ, say by edges $e_{i_1}, \dots e_{i_p}$.

Then $d(e_{i_1} + \dots + e_{i_p}) = V_i - V_\phi$.

Now, suppose we have $d(e_i) = v_k - v_j$.
If $v_k = v_1$, then $v_k - v_j = -(v_j - v_1)$.

If $v_k \neq v_1$, then $v_k - v_j = v_k - v_1 - (v_j - v_1)$.

Hence, it follows that $d(\sum r_i e_i) = \sum_{j=2}^{n} r_j'(v_j - v_1)$

Hence $\operatorname{im} d = \operatorname{span}(v_2 - v_1, \ldots, v_n - v_1)$. Hence it follows that $H_0(C_\bullet) = Rv_1$ is free of rank 1.

Now, consider $R = \mathbb{Z}$. Hence $\ker d$ is free of rank $m-n+1$ and so the general case follows. $\blacksquare$

Snake lemma: If we have exact rows:

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' & \end{tikzcd}$$

then exists long exact sequence:

$$\ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h$$

Question 2

we have

$$\begin{tikzcd} A_n/B_n(A_\bullet) \arrow[r] \arrow[d, "d"] & B_n/B_n(B_\bullet) \arrow[r] \arrow[d, "d"] & C_n/B_n(C_\bullet) \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & Z_{n-1}(A_\bullet) \arrow[r] & Z_{n-1}(B_\bullet) \arrow[r] & Z_{n-1}(C_\bullet) \end{tikzcd}$$

if the rows are exact, then snake lemma gives us

$$H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(C_\bullet) \to H_{n-1}(A_\bullet) \to H_{n-1}(B_\bullet) \to H_{n-1}(C_\bullet).$$

which is what we want. we can show the rows are exact via snake lemma.

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r] \arrow[d, "d"] & B_n \arrow[r] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0 \end{tikzcd}$$

gives $0 \to Z_n(A_\bullet) \to Z_n(B_\bullet) \to Z_n(C_\bullet) \to A_{n-1}/B_{n-1}(A_\bullet) \to B_{n-1}/B_{n-1}(B_\bullet) \to C_{n-1}/B_{n-1}(C_\bullet) \to 0$

exact.

Tor functors.

- For R-module M, a free resolution is an exact sequence

$$\cdots \longrightarrow R^{\oplus s_2} \longrightarrow R^{\oplus s_1} \longrightarrow R^{\oplus s_0} \longrightarrow M \longrightarrow 0$$

throw M away, and tensor with R-module N. to obtain chain complex $M_\bullet \otimes_R N$:

$$\cdots \longrightarrow N^{\oplus s_2} \longrightarrow N^{\oplus s_1} \longrightarrow N^{\oplus s_0} \longrightarrow 0.$$

(tensors commute with colimits as left adjoints)

define $\mathrm{Tor}_i^R(M,N) := H_i(M_\bullet \otimes N)$

- this doesn't depend on choice of free resolution.

Question 3:

we have $N^{\oplus s_2} \to N^{\oplus s_2} \to M \otimes_R N \to 0$ exact since $- \otimes_R N$ is right exact.

Hence $\mathrm{Tor}_0^R(M,N) = M \otimes_R N$.

Q4.

Essentially, we have

$$\begin{tikzcd}
0 \arrow[r] & R^{\oplus S'_\bullet} \arrow[r] \arrow[d] & R^{\oplus S_\bullet} \arrow[r] \arrow[d] & R^{\oplus S''_\bullet} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

after apply $-\otimes N$, the rows of the resolution stay exact since free modules are projective. This gives us a SES

$$0 \to N^{\oplus S'_\bullet} \to N^{\oplus S_\bullet} \to N^{\oplus S''_\bullet} \to 0$$

and the corresponding LES in homology is what we want.

Question 5.

submodules of a free module are free over a PID

Hence we have a resolution

$$0 \to R^{\oplus i} \to R^{\oplus j} \to M \to 0$$

and the result follows

Question 6

$$0 \to (r) \to R \to R/(r) \to 0 \quad (\text{as } R\text{-modules})$$

gives us

$$\mathrm{Tor}_1^R(R,N) \to \mathrm{Tor}_1^R(R/(r),N) \to (r)\otimes_R N \to$$

$$\to R\otimes_R N \to R/(r) \otimes_R N \to 0$$

Note, $\mathrm{Tor}_1^R(R,N)=0$, and so $\mathrm{Tor}_1^R(R/(r),N)$ is isomorphic to $\ker((r)\otimes_R N \to R\otimes_R N)$ which is exactly the r-torsion of n. $\boxtimes$