Week 5 Notes.

Lemma (From H/W):
Given I an ideal of Ring R. Then for any R-module M we have
$$R/I \otimes_R M \cong M/IM.$$

Proof:
We have SES: of R-modules

$$0 \to I \to R \to R/I \to 0$$

applying the right exact functor $- \otimes_R M$ gives

$$I \otimes_R M \to R \otimes_R M \to R/I \otimes_R M \to 0$$

exact. Hence as $I \otimes_R M \cong IM$ and $R \otimes_R M \cong M$ (and these iso agree). It follows that
$$R/I \otimes_R M \cong M/IM.$$
$\blacksquare$

Question 1: Take $M = R/J$ and use the third isomorphism theorem.

$$R/I \otimes_R R/J \cong (R/J)/(I(R/J))$$

$$= (R/J)/((I+J)/J)$$

$$\cong R/I+J.$$
$\blacksquare$

Question 2:

① $(S^{-1}A) \otimes_R M \cong S^{-1}(A \otimes_R M)$

② $(S^{-1}M) \otimes_{S^{-1}R} N \cong M \otimes_R N$.

Proof ①.

- can be "checked by hand"
- could be proven via universal property of localisations.
- Simplest, use associativity:

$$(S^{-1}A) \otimes_R M \cong (S^{-1}R \otimes_R A) \otimes_R M$$

$$\cong (S^{-1}R) \otimes_R (A \otimes_R M)$$

$$\cong S^{-1}(A \otimes_R M).$$

Proof ②: Again, can be done by hand, but associativity to the rescue.

$$(S^{-1}M) \otimes_{S^{-1}R} N \cong (M \otimes_R S^{-1}R) \otimes_{S^{-1}R} N$$

$$\cong M \otimes_R (S^{-1}R \otimes_{S^{-1}R} N)$$

$$\cong M \otimes_R N.$$

Question 3

M flat $\Rightarrow$ M_p flat by previous Q sing

if $0 \to N' \to N \to N'' \to 0$ SES of R_p-modules. Then

$$0 \to M_p \otimes_{R_p} N' \longrightarrow M_p \otimes_{R_p} N \longrightarrow M_p \otimes_{R_p} N'' \to 0$$

is a sequence, which is isomophic to the following

$$0 \longrightarrow M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \longrightarrow 0$$

which is exact as M flat.

Conversely, if all M_p flat for all primes P and suppose we have a SES of R-modules

$$0 \to N' \longrightarrow N \to N'' \to 0$$

then we have a sequence

$$0 \to M \otimes_R N' \to M \otimes_R N \longrightarrow M \otimes_R N'' \to 0$$

which we just need to show that the map $M \otimes_R N' \to M \otimes_R N$ is injective

From last week, injectivity is a local property and by previous question we have that

$$(M \otimes_R N')_p \to (M \otimes_R N)_p$$

is equivalent to

$$M_p \otimes_R N' \to M_p \otimes_R N$$

and as M_p flat, we are done. $\boxtimes$

Question 4

(1) $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong (\mathbb{Q}_{(0)}) \otimes_{\mathbb{Z}_{(0)}} \mathbb{Q}$

$\cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ by Q2.

(2) $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ which is dimension 2 over $\mathbb{R}$.

while $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has dimension 4. over $\mathbb{R}$.