Theorem 1 proof:

let x_i be the generators of M. Then
$f(x_i) = \sum a_{ji} x_i$ for some $a_{ij} \in I$

Let $A = (a_{ij}) \in M_n(I)$. This is such that if $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ then $f(\vec{x}) = A\vec{x}$. We have $A[f]$

commutative, so $(fI - A)\vec{x} = 0$. Let $B = fI - A$
then $\operatorname{adj}(B)B = \det B$ and so it follows that
$\det B I \vec{x} = 0 \Rightarrow \det B\, m = 0$ for all $m \in M$.
Moreover, $\det B = f^n + a_{n-1} f^{n-1} + \cdots + a_0$ for some $a_i \in I$.

Question 1

1) Just take $f = 1$ in above theorem.

2) we have an element $x \in I \subseteq Rad(R)$ such that $xm = m$ for all $m \in M$. $1 - x$ is a unit (Jacobson rad) and so must act as an isomorphism on M.
but $(1-x)m = 0$. Hence $M = 0$.

3) Apply the previous to M/N.

4) Let $N = \sum R x_i$. Then as local ring, $m = rad(R)$

and so $M = mM + N$ if the x_i form an F-basis.

Hence by previous, $N = M$.

Question 2

If R not a field, exists maximal ideal m

Suppose F is a f.g. R-module, then it is also finitely generated as a R_m-module

Since R_m is a local ring with maximal ideal m_m, $m_m = Rad(R_m)$ and we also clearly have

$m_m F = F$. Hence by Nakayama, $F = 0$ a contradiction

Question 3

Let $f: M \to M$ be surjective. $R[f]$ then acts on M and as f surjective, $(f)M = M$. Hence, there exists an element $x \in (f)$ such that $xm = m$ for all $m \in M$. Hence we conclude that f is an isomorphism.

Question 4

Note that $S^{-1}M \cong S^{-1}R \otimes_R M$ and so the functor is right exact since the tensor functor is right exact. Hence we just need to show it preserves injectivity.

Suppose $\varphi: M \to N$ injective R-hom

Then $S^{-1}(\varphi): S^{-1}M \to S^{-1}N$ is given by

$S^{-1}(\varphi)(m/s) = \varphi(m)/s$. If $\varphi(m)/s = 0$, then there

exists $t \in S$ s.t $t\varphi(m) = 0$ ie $tm = 0$ since φ injective

and so $m/s = 0$. Hence $S^{-1}(\varphi)$ injective and

S^{-1} is an exact functor

Question 5

① If $M=0$, then $M_p=0$ for prime p trivially.

Suppose $M\neq 0$. Let $m\neq 0$ in M and consider the ideal $\mathrm{Ann}(m)$. Since $a\in \mathrm{Ann}(m)$, this is nonempty and by Zorn's, there exists a maximal ideal $\mathfrak{m}$ that contains it.

Suppose $M_{\mathfrak{m}}=0$. Then there must exist $r\notin \mathfrak{m}$ such that $rm=0$ which is a contradiction. Hence $M_{\mathfrak{m}}\neq 0$.

② We previously showed that if $\varphi: M\to N$ injective then $\varphi_p: M_p\to N_p$ injective for all prime p.

Conversely, suppose φ_p injective for all prime p. We have the exact sequence

$$0\to \mathrm{Ker}(\varphi)\to M\xrightarrow{\varphi} N\to N/M\to 0.$$

Applying the functor, we get

$$\operatorname{Ker}(\varphi)_p \to M_p \xrightarrow{\varphi_p} N_p$$

exact for all prime p, and as injective, we have $\ker(\varphi)_p = 0$ for all p and by previous $\ker(\varphi) = 0$.

③ Let $\operatorname{Tor}(M)$ be the submodule of M of torsion elements.

It is not hard to show $\operatorname{Tor}(M_p) = \operatorname{Tor}(M)_p$ and so this follows from ①.