Week 3

Question 1

(1) Let $i: R \to R[S^{-1}]$ be the map $r \mapsto r/1$. Given an ideal $I \subseteq R$, let $\bar{I} \subseteq R[S^{-1}]$ be the ideal generated by $i(I)$. Observe that all elements of $\bar{I}$ can be written as r/s where $r \in I$.

Now, consider the map $\Phi: \{\text{primes } P \subseteq R, P \cap S = \phi\} \to \{\text{primes of } R[S^{-1}]\}$ given by $\Phi(P) = \bar{P}$.

Given prime P, $\bar{P}$ is also prime since if $\frac{a}{r} \cdot \frac{b}{s} \in \bar{P}$ then $ab/rs = p/q$ where $p \in P$ and so exists $s' \in S$ such that $s'(qab - rsp) = 0$ and so $s'qab \in P$. Since $s', q \in S$ and $P \cap S = \phi \Rightarrow a$ or $b \in P$ and so either a/r, or $b/s \in \bar{P}$.

If $\bar{P} = R[S^{-1}]$, then $1/1 \in \bar{P}$ and so $1/1 = p/q$ where $p \in P$ and exists $s \in S$ s.t $s(q-p) = 0$ and $sq \in P$. $\Rightarrow s$ or $q \in P$ i.e $P \cap S \neq \phi$ which is a contradiction.

Hence Φ is well defined. It has an inverse given by $\Psi(Q) = i^{-1}(Q)$. If Q is a prime ideal in $R[S^{-1}]$, suppose $ab \in i^{-1}(Q)$. Then $\frac{ab}{1} \in Q$ and so either $a/1$, or $b/1 \in Q$ and so either a or b in $i^{-1}(Q)$. Moreover, since it's a preimage of an ideal, it's an ideal. Hence Ψ is well defined. $(i^{-1}(Q) \cap S = \phi$

by above argument)

It is not hard to show that $\Phi\Psi = id_{R[S^{-1}]}$ and

$\Psi\Phi = id_R$.

(2) You can either use the universal property or show directly that

$$(R[S^{-1}])(\bar{W}^{-1}) \longrightarrow R[W^{-1}]$$
$$\frac{a/s}{w/1} \longmapsto \frac{a}{sw}$$

is an isomorphism.

Question 2

g doesn't vanish on $D(f)$ if for all $[p] \in D(f)$, $g \notin P$. ie if $f \notin P$ then $g \notin P$. In particular, $f \in S$. Hence by the universal property we have a natural map $R_f \to R[S^{-1}]$. We need to show that for all $g \in S$, g is a unit in R_f.

Now, $D(g) \supseteq D(f) \Rightarrow V(g) \subseteq V(f)$
$\Rightarrow IV(g) \supseteq IV(f)$

ie $\sqrt{(g)} \supseteq \sqrt{(f)}$ and so exists n st $f^n = hg$ for some n. Hence g is a unit in R_f

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Question 3

Observe that we have $D(f) \cap D(g) = D(fg)$.

Proof: $[p] \in D(f) \cap D(g) \Leftrightarrow f \notin p$ and $g \notin p$.
$\Leftrightarrow fg \notin p$ since prime
$\Leftrightarrow [p] \in D(fg)$ $\blacksquare$

Moreover, $X = \operatorname{Spec} R$ is covered by base elements as $D(1) = X$.
Suppose $U = X \setminus V(I)$ and $[p] \in U$.
Then as $[p] \notin V(I)$, there exists $f \in I$ s.t. $f([p]) \neq 0$
Then $V(f) \supseteq V(I) \Rightarrow D(f) \subseteq U$ and $[p] \in D(f)$.

Hence the basic open sets $D(f)$ form a basis for X

Now, suppose $X = \bigcup_{i \in I} D(f_i)$ is covered by basic open sets
Then $X = X \setminus \bigcap V(f_i) = X \setminus V((f_i)_{i \in I})$

So no prime ideal contains $(f_i)_{i \in I}$

Hence this is the whole ring and $1 = \sum_{i=1}^{n} h_i f_i$

for some $h_i \in R$. Then $X = \bigcup_{i=1}^{n} D(f_i)$ and so

compact.

Question 4

(1) It is sufficient to show that if $g=0$ in R_{f_i} for all f_i, then $g=0$ in R.

If $g=0$ in R_{f_i}, then exists n_i such that $f_i^{n_i} g=0$ in R.

Since $\cup D(f_i) = X \,(=\operatorname{Spec} R)$, then $1 = \sum h_i f_i$ for some h_i. Let $n = \sum n_i$. Then $1^n = (\sum h_i f_i)^n$ is such that $(\sum h_i f_i)^n g = 0$ ie $1 \cdot g = 0 \Rightarrow g=0$.

(2) if $g_i \in R_{f_i}$, then $g_i = g_i'/f_i^{n_i}$ for some $g_i' \in R$.

Moreover, if $g_i = g_j$ in $R_{f_i f_j}$, Then

$\cancel{\frac{g_i' f_j^{n_j}}{f_i^{n_i} f_j^{n_j}} = \frac{g_j' f_i^{n_i}}{f_j^{n_j} f_i^{n_i}}}$ ~~and so exists~~ $\cancel{m}$ ~~such that~~

$$\cancel{(f_i f_j)^m (g_i' f_j^{n_j} - g_j' f_i^{n_i}) = 0 \text{ in } R}$$

$$\cancel{g_i' f_i^m f_j^{n_j+m} = g_j' f_j^m f_i^{m+n_i}}$$

$$\frac{g_i' f_j^{n_j}}{f_i^{n_i} f_j^{n_j}} = \frac{g_j' f_i^{n_i}}{f_i^{n_i} f_j^{n_j}}$$

Let $h_i = f_i^{n_i}$ then $\frac{g_i' h_j}{h_i h_j} = \frac{g_j' h_i}{h_i h_j}$

so there exists m, $(h_i h_j)^m (g_i' h_j - g_j' h_i) = 0$.

Note that $D(f_i) = D(h_i)$. (for later).

So $g_i' h_i^m h_j^{m+1} = g_j' h_j^m h_i^{m+1}$

let $g_i'' = g_i' h_i^m$ and $k_i = h_i^{m+1}$.

Then $D(k_i) = D(h_i) = D(f_i)$ and we have

$$g_i'' k_j = g_j'' k_i \qquad (*)$$

Since $X = \bigcup_{i=1}^{n} D(k_i)$ we have $1 = \sum r_i k_i$ for some $r_i \in R$.

Let $r = \sum r_i g_i''$

Note that
$$r k_i = \Big(\sum_j r_j g_j''\Big) k_i$$
$$= \Big(\sum_j r_j k_j\Big) g_i'' \quad \text{by } (*)$$
$$= g_i''$$

Hence, in $D(f_i)$, since $k_i = f_i^{n_i m}$

in R_{f_i} we have $r = g_i''/k_i = g_i' h_i^m / h_i^{m+1}$

$= g_i'/f^{n_i} = g_i$. Hence we are done.