MATH 210B

Question 1:

We will prove this as follows: Let $p \in R$ be a prime element. Then $R/(p)$ is a domain and $\bar{f}, \bar{g} \neq 0$ in $R/(p)[x]$ since primitive. Hence $\bar{f}\bar{g} \neq 0$ in $R/(p)[x]$ and so p does not divide all coefficients of fg. As p arbitrary prime then fg is primitive.

Question 2:

I'll skip well define-ness as that's annoying.

Identities are $0/1$, $1/1$.

Let $i: R \to F$ be the map $i(a) = a/1$. Now, suppose that $i(a) = 0/1$ so $a/1 = 0/1$. Then $1 \cdot a = 0 \cdot 1 \Rightarrow a = 0$. Hence i injective.

Question 3:

Observe the following: if we have $f = cg$ where $f, g \in R[x]$, $c \in F$ and g primitive, then we must have $c \in R$. This follows as let $c = a/b$, $a, b \in R$. Then $bf = ag \Rightarrow b \cdot \mathrm{cont}(f) = a$ as g primitive and so $a/b = \mathrm{cont}(f) \in R$.

Now, onto the question:

($\Rightarrow$) If f nonconstant irreducible in $R[x]$, suppose $f = gh$ where $g, h \in F[x]$. By grouping denominators and factoring coefficients we get $f = cg'h'$ where $c \in F$, $g', h' \in R[x]$ and primitive. So $g'h'$ primitive and we conclude $c \in R$. Since f primitive, c is a unit. This contradicts irred. of f in $R[x]$

($\Leftarrow$) clear.

Question 4

Note: $F[x]$ is PID so UFD.

we factor $f \in R[x]$ in $F[x]$ and by previous observations we can do this as

$$f = cf_1'f_2' \cdots f_n'$$

where $c \in R$ and $f_i' \in R[x]$ and primitive and irreducible. Since R is UFD, this tells us we can factor in $R[x]$. Uniqueness follows a standard arguement using uniqueness in R and $F[x]$.

Question 5

No. Consider $R = k[x_1, x_2, \dots]$.

This is not noetherian but is a UFD.

(each element belongs to some subring $k[x_1, \dots, x_n]$ which is a UFD.

Question 6:

WLOG, assume f is primitive. So by Gauss, we want to show that f irreducible in $R[x]$. Suppose this wasn't the case. Then $f = gh$ where $g, h \in R[x]$ and primitive as well as nonconstant.

Now, in $R/(p)[x]$ we have that $\bar{f} = \bar{a}_n x^n = \bar{g}\bar{h}$ so the constant terms of g and h are divisible by p but this contradicts $p^2 \nmid a_0$.

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