Q1 a) $V(r(\underline{a})) \subseteq V(\underline{a}) \subseteq V(E)$ clear.

If P prime such that $E \subseteq P$ then $\underline{a}E \subseteq \underline{a}P = P$ so $\underline{a} \subseteq P$. Also we must have $\sqrt{\underline{a}} \subseteq P$ since suppose $x \in \sqrt{\underline{a}}$. Then $x^n \in \underline{a}$ for some n. Then $x^n \in P$ $\Rightarrow x \in P$ as prime. Hence $V(E) \subseteq V(\sqrt{\underline{a}})$. ∎

b) Every ideal contains 0, so $\mathrm{Spec}(R) = V(0)$. Since prime ideals are proper, none contain 1. $V(1) = \emptyset$.

c) $P \in V(\cup E_i) \Leftrightarrow \cup E_i \subseteq P$
$\Leftrightarrow E_i \subseteq P$ for all i
$\Leftrightarrow P \in V(E_i)$ for all i
$\Leftrightarrow P \in \cap V(E_i)$

d) $V(\underline{a}) \cup V(\underline{b}) \subseteq V(\underline{a} \cap \underline{b})$ clear.

Suppose $P \in V(\underline{a} \cap \underline{b})$ so $\underline{a} \cap \underline{b} \subseteq P$ and suppose $\underline{a} \not\subseteq P$ and $\underline{b} \not\subseteq P$. Let $x \in \underline{a} \setminus P$, $y \in \underline{b} \setminus P$. Then $xy \in \underline{a} \cap \underline{b}$ so $xy \in P \Rightarrow x \in P$ or $y \in P$ a contradiction.

Q2 If f nilpotent, then let $[p] \in \operatorname{Spec} R$. Since $f^n = 0 \in P$ for some n, $f \in P$ and so $f([p]) = 0$. Hence everywhere zero.

Conversely, suppose f is not nilpotent so the suggested set S does not intersect 0.

Let I be the set of all ideals that don't intersect S. It is non empty as (0) is an ideal and every chain has an upper bound under inclusion by taking unions. Hence I contains a maximal element by Zorn.

Let m be this element. We use ideal quotients here. Suppose $xy \in m$ but $x, y \notin m$. Then $(m;x) \supseteq m$ and larger. Hence $f^n \in (m;x)$ for some n. Similarly, $(m; f^n) \supseteq m$ and larger. Hence, exists m such that $f^{n+m} \in m$ a contradiction

$\boxed{5}$

Q3 $\mathrm{Spec}(R/\underline{a})$ is the subspace $V(\underline{a}) \subseteq \mathrm{Spec}(R)$.

This follows via bijection between ideals and quotients.

- $\mathrm{Spec}(R) = \mathrm{Spec}(R/N)$

Q4 It is enough to show that $IV(J) = \sqrt{J}$.
That is, the intersection of all primes that contain J is it's radical. This follows from Q2 and the bijection between ideals after quotienting by J.

Now, let C be a closed set of spec R.
so $C = V(I)$ for some I. Then we have
$$IVI(C) = IVIV(I) = IV(\sqrt{I}) = I(C).$$

so $\quad IVI(C) = I(C)$

and $\quad VIV(J) = V(J)$ for radical J.

This gives us a bijection as required. Note $I(C)$ is radical, $V(J)$ closed so IV and VI identity.

Note: R not nec. commutative in this Q.

Q5 If $R = \bigoplus_{i=1}^{n} R_i$ then let e_i be the identity of R_i.

Then e_i are the central idempotents that sum to 1.

Conversely, we let $R_i = Re_i$ and since e_i central, this is a subring. Let $\bigoplus R_i \to R$ be the map determined by the inclusions $R_i \hookrightarrow R$ (sum is coproduct) since $\sum e_i = 1$, this is easily seen to be surjective. Note that if $re_i = se_j$ $i \neq j$, then apply e_j on left gives $re_ie_j = se_j^2$ $\Rightarrow 0 = se_j$ as orthogonal. Essentially the same argument shows that $(R_1 + R_2 + \cdots + R_{i-1} + R_{i+1} + \cdots + R_n) \cap R_i = 0$ for all i and so $\bigoplus R_i \to R$ is injective.

$\blacksquare$