Math 210A
Midterm Exam II
November 15, 2019

Name: Ben Szczesny

Let me know if you find an error. Also, there are likely more ways to do these questions and these are likely not the answers Prof. Merkurjev had in mind.

1	2	3	4	5	Σ

---

1. Let P be a Sylow 2-subgroup of the symmetric group S_n with n odd. Prove that there is a symbol $i \in \{1, 2, \ldots, n\}$ such that $\sigma(i) = i$ for all $\sigma \in P$.

Let $\mathfrak{X} = \{1, 2, \ldots, n\}$. Since P is a permutation group, we have a natural action $P \curvearrowright \mathfrak{X}$.

From Orbit-Stabalizer, each orbit has size 2^k for some $k \in \mathbb{Z}_{\geq 0}$ and so

$$|\mathfrak{X}| \equiv |\mathfrak{X}^P| \pmod 2$$

Since $|\mathfrak{X}|$ odd, $|\mathfrak{X}^P| \neq 0$ and so then is some point $x \in \mathfrak{X}$ such that $\sigma(x) = x$ for all $\sigma \in P$. $\blacksquare$

2. Prove that for every group G the (external) semidirect product $G \rtimes G$ with respect to the homomorphism $G \to Aut(G)$ taking an element g to the inner automorphism of G given by g, is isomorphic to the (external) direct product $G \times G$.

We will construct a left split exact sequence

$$1 \to G \xrightarrow{\varphi} G \rtimes G \xrightarrow{\psi} G \to 1$$

which then implies $G \rtimes G \cong G \times G$.

Consider the map $\psi : G \rtimes G \to G$ given by $(g,h) \mapsto gh$. This is easily shown to be an epimorphism. Now, consider $\varphi : G \to G \rtimes G$ given by $\varphi(g) = (g, g^{-1})$.

Then
$$\begin{aligned} \varphi(g)\varphi(h) &= (g, g^{-1})(h, h^{-1}) \\ &= (gg^{-1}hg, g^{-1}h^{-1}) \\ &= \varphi(hg) \end{aligned}$$

So this is a hom $\varphi : G^{op} \to G \rtimes G$ and $\operatorname{Im} \varphi = \operatorname{Ker} \psi$. Since $G^{op} \cong G$ (via inversion) we get a short exact sequence as we wanted. Now, consider $p : G \rtimes G \to G$ given by $(g,h) \mapsto g$. Then $p\varphi = id_G$ and so this sequence splits and we are done.

3. Let $1 \to H \xrightarrow{i} G \xrightarrow{f} F \to 1$ be a short exact sequence of groups such that F is a cyclic group of order n and H is a group of order m. Prove that if n and m are relatively prime, then the exact sequence is split.

We have $F = \langle x \rangle$ for some x, let $y \in f^{-1}(x)$, then $\operatorname{ord}(y) = n$ and $z = y^m$ is also such that $\operatorname{ord}(z) = n$ as $\gcd(n, m) = 1$.

Let $\varphi : F \to G$ be given by $\varphi(x) = z$. Since $z^n = 1$, this is a well defined hom. Moreover, $f\varphi(x) = x \Rightarrow f \circ \varphi = id_F$ and hence the above SES is right split.

Note, we don't get left split in general, consider

$$1 \to \mathbb{Z}/3\mathbb{Z} \to \underbrace{\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}}_{\text{is } S_3} \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

4. Prove that every group generated by two elements of order 2 is solvable.

Suppose G is generated by x, y s.t $x^2 = y^2 = e$.

Then since $G/\langle\langle [x,y] \rangle\rangle$ is abelian we have $[G,G] = \langle\langle [x,y] \rangle\rangle$.

I claim $\langle\langle [x,y] \rangle\rangle = \langle xy \rangle$.

This can be seen by the following calculations.

$$[x,y] = (xy)^2 \qquad y[x,y]y = [y,x]$$

$$x[x,y]x = [y,x] \qquad x[x,y]y = (xy)$$

Hence $[G,G]$ is cyclic and so abelian.

Therefore we have a finite derived series

$$G \trianglerighteq [G,G] \trianglerighteq \{e\}.$$

and G is solvable

5. Prove that the free group $F_n := F(\{1, 2, \ldots, n\})$ cannot be generated by less than n elements.

Suppose F_n can be generated by m elements.

Then we have an epimorphism $F_m \overset{\varphi}{\twoheadrightarrow} F_n$ and if we let $S = \langle w^2 \mid w \in F_m \rangle$, this is normal in F_m and so we get an epimorphism

$$\begin{array}{ccc} F_m/S & \overset{\tilde{\varphi}}{\twoheadrightarrow} & F_n/\varphi(S) \\ \wr\Vert & & \wr\Vert \\ (\mathbb{Z}/2\mathbb{Z})^m & & (\mathbb{Z}/2\mathbb{Z})^n \end{array}$$

but such an epimorphism exists only if $m \geq n$. ∎