MATH210A: Week 7

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Question 1. Let G be a finite group and $H < G$ a proper subgroup. Show that $\cup_{g \in G} gHg^{-1} \neq G$. Bonus question: What happens when G is infinite?

Question 2. Show that for a finite group G and proper subgroup H, there exists a conjugacy class of G that does not intersect H.

Question 3. (Useful results for counting) Let G be a group and X a G-set. For $x, y \in G$, show that

(a) if x and y are conjugate then $|X^x| = |X^y|$.

(b) if x and y generate the same subgroup then $|X^x| = |X^y|$.

Question 4. Use Burnside's lemma to answer the following counting problem. Let n be an even number and and suppose we have n indistinguishable balls and put them into 3 indistinguishable jars. How many ways can we do this?

Bonus questions

Note, you don't need burnside to do these questions.

Question 5.

(a) Let X be a finite G-set with $|G| = p^n$ for some prime p and p does not divide $|X|$. Show there exists an element $x \in X$ such that $gx = x$ for all $g \in G$.

(b) Let V be a d-dimensional vector space over $\mathbb{Z}_p$ and let $G \subset GL_d(\mathbb{Z}_p)$ be a group such that $|G| = p^n$. Show that there exists a nonzero vector $v \in V$ such that $g \cdot v = v$ for all $g \in G$.

Question 6. Prove the Frattini argument: Let G be a finite group and $H \triangleleft G$. Suppose P is a Sylow p-subgroup of H. Then $G = HN_G(P)$.

A word on homework

This is a grad level course so your presentation of solutions is more important than what was expected in undergrad (I also don't have time to spend 15 minutes dissecting an answer). Here are a few things things to keep in mind when writing your future homework assignments:

(a) Presentation is important. This means using white space and paragraphs. When your argument is long, break it into parts (You can use subclaims/lemmas of your own making!)

(b) You do not need to include routine calculations. For instance, if you claim something is a homomorphism and the proof of which could be done in a few lines of calculations, then I believe you. You don't need to include these calculations.

(c) Less is more. Almost all of these homework questions can be done in less than a page. If your argument is longer, than think about how to simplify it. I care more for a clear argument than whether you checked every detail. Again, this is a grad class, so I assume everyone is capable of checking the small details.