MATH210A: Week 6

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Crash course on representation theory

Question 1. For each of the following representations $\rho : G \to GL(V)$, describe what the matrices $\rho(g)$ look like.

(a) V is a one dimensional vector space. Note, G is finite.

(b) Let X be a G-set. Then the action of G on X extends linearly to an action on $F(X)$. Take $V = F(X)$ and $\rho(g)$ is given by $\rho(g)(x) = g \cdot x$ on the basis $x \in X$.

Question 2. We will prove Maschke's theorem: Let k be a field such that $|G|$ does not divide the order of k. Then any k-representation $\rho : G \to GL(V)$ is completely reducible. (If you don't know what characteristic is, take k to be $\mathbb{R}$ or $\mathbb{C}$)

(a) Let W be a G-invariant subspace of V and $\pi : V \to V$ any projection onto W. Define the following map $\pi' : V \to W$ by
$$\pi'(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)\pi(\rho(g^{-1})v).$$
Show that π' is also a projection onto W. A projection on W is a linear transformation such that $\pi^2 = \pi$ and $\mathrm{im}(\pi) = W$.

(b) Show that π' is a G-linear map.

(c) Show that there exists a G-invariant subspace W' such that $W \oplus W' = V$. That is, V is completely reducible.

Question 3. Let $G = \{1, x, x^2\}$ be the cyclic group of order 3 and define a complex representation $\rho : G \to GL(\mathbb{C}^3)$ by $\rho(x)(z_1, z_2, z_3) = (z_2, z_3, z_1)$. Find the irreducible G-invariant subspaces of $\mathbb{C}^3$. (There will only be three of them for reasons, what's special about linear operators over complex numbers?)

Question 4. We will prove Schur's lemma (Some version of it at least). Consider a irreducible complex representation $\rho : G \to V$. Let $\phi : V \to V$ be a G-linear map. Show there exists a scalar $\lambda \in \mathbb{C}$ such that $\phi(v) = \lambda v$ for all $v \in V$.

Question 5. Let G be an abelian group. Prove all irreducible complex representations of G are one-dimensional.