MATH210A: Week 5

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Finitely Generated Abelian Groups

Let G be a finitely generated abelian group. We then have two forms of classification:
Primary Decomposition Version:

$$G \cong \mathbb{Z}^n \oplus \mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_r}$$

where n is called the rank and the q_i are prime powers (not necessarily distinct primes) call the elementary divisors. Moreover, the values $n, q_1, \cdots q_r$ are uniquely determined by G.
Invariant Factor Version:

$$G \cong \mathbb{Z}^n \oplus \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_r}$$

where n is the rank and the a_i are positive integers called the invariant factors and are such that $a_i | a_{i+1}$. Moreover, the values $n, a_1, \cdots, a_r$ are uniquely determined by G.

Question 1. Let $\mathbb{F}$ be a field and H some finite subgroup of $\mathbb{F}^\times$. Show that H is cyclic.

Question 2. Show that a finite abelian group G is generated by its elements of maximal order.

Question 3. Show that the invariant factors of $\mathbb{Z}_n \oplus \mathbb{Z}_m$ are $gcd(m, n), lcm(m, n)$ is $gcd(m, n) > 1$ and mn if $gcd(m, n) = 1$.

Question 4. Let G be a finite abelian group of order n. Show that if G has at most one subgroup of order d for every divisor d of n, then G is cyclic.

Misc.

Question 5. (Summer '16 Q9) Suppose G is a finite group that acts transitively on a set X with at least two elements. Show that there is an element of G that fixes no element of X.

Question 6. (Fall '11 Q2) Let G be a non trivial finite group, and p a prime. Suppose each proper subgroup H is such that the index $[G : H]$ is divisble by p. Show that the center $Z(G)$ is has order divisible by p.

Question 7. (One of my personal favourite riddles)

Alice and Bob are on death row and the Warden, a sadistic man, one day calls them in to his office to play a game of luck. If they win, he will set them free. If they lose, he will feed them to his pet sharks. The game is as follows, both Alice and Bob will be sent outside his office while he shuffles a deck of 52 playing cards. He will then call Alice in to examine the deck and she is allowed to swap exactly two cards, if she so chooses. The Warden then will make Alice exit through a different door so she can not talk to Bob. The Warden will then call Bob into his office and spread the deck face down on his desk in order. The Warden will then name a card and Bob then has 26 guesses to pick the correct card. The Warden then sends both Alice and Bob outside while he shuffles the deck. They talk and 10 minutes later they are free. Why?