MATH210A: Week 4

TA: Ben Szczesny Last updated: 2019/10/21

Nilpotent and Solvable Groups

We will assume that all groups G are finite. There are a bunch of equivalent definitions for nilpotent groups, but we will use this one.

Definition. A group G is nilpotent if it has a finite central series. That is, a normal series

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

such that $G_{i+1}/G_i \leq Z(G/G_i)$ or equivalently $[G, G_{i+1}] \leq G_i$.

Note, a normal series of G is one where each subgroup is normal in G. Compare this to a subnormal series where they are just normal in the next consecutive group. We have the following theorem

Theorem 1. *The following a equivalent*

(a) Group G is nilpotent

(b) If H is a proper subgroup of G, then H is a proper subgroup of $N_G(H)$. That is, normalizers grow.

(c) Every Sylow subgroup is normal.

(d) G is the direct product of Sylow subgroups.

(e) If d divides $|G|$, then G contains a normal subgroup of order d.

While we won't go through the proof of this, it's a good exercise to prove this theorem.

Question 1. (Why the heck are they called nilpotent groups anyway?) Consider the map $\text{adj}_g : G \to G$ given by $\text{adj}_g(x) = [g, x]$. This is called the adjoint map. Show that if G is nilpotent, then there exists some n such that $\text{adj}_g^n(x) = e$ for all $x, g \in G$.

Question 2. Consider the quarternion group

$$Q_8 = \langle -1, i, j, k \mid i^2 = j^2 = k^2 = ijk = -1, (-1)^2 = 1 \rangle.$$

Is this nilpotent?

Question 3. Prove that G is nilpotent if and only if $G/Z(G)$ is nilpotent.

Question 4. Prove that the dihedral group

$$D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$$

is nilpotent if and only if $n = 2^k$ for $k \in \mathbb{N}$.

Definition. A group G is solvable if it has a subnormal series with each factor group abelian.

Question 5. The derived series of a group G is the normal series

$$G = G^{(0)} \triangleright G^{(1)} \triangleright \cdots$$

where $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. Show that a group is solvable if and only if the derived series eventually terminates at $\{e\}$.

Question 6. Show that nilpotent groups are solvable. Can you think of an example of a solvable group that is not nilpotent?

Question 7. Show that all groups of order < 60 are solvable. *Hint: A_5 is the smallest simple nonabelian group.*