MATH210A: Week 1 Problems

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Question 1. (Warm up) Suppose group G acts on finite set X. Given $x \in X$ show that:

(a) The Stabalizer subgroup G_x is a subgroup.

(b) The orbit equivalence is an equivalence relation. i.e, we have for $x, y \in X$ that $x \cong y$ if there exists $g \in G$ such that $g \cdot x = y$.

Question 2. Prove the Orbit-Stabalizer theorem: Suppose group G acts on finite set X. Then for any $x \in X$ we have that $|G| = |G \cdot x||G_x|$.

Question 3. Suppose group G acts on finite set X. Denote by $|X/G|$ to be the number of orbits under the action, and for $g \in G$ we denote X^g to be the set fixed by g. Prove Burnside's lemma:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

Question 4. Prove the class equation:

$$|G| = |Z(G)| + \sum_i |G : C_G(x_i)|$$

where the sum is over representative elements of each conjugacy class not in the center.

Question 5. (Useful lemma for the Sylow theorems) Let p be a prime and G a group with $|G| = p^n$. Show that if G acts on finite set X, then $|X| = |X^G| \mod p$.

Question 6. Prove Cauchy's theorem: Let G be a finite group and p a prime such that $p||G|$. Then there exists a subgroup of G of order p.