Week 9 notes

(Some of these are examples I found in the without loss of universality blog)

Yoneda Lemma:

Let $\mathcal{C}$ be a locally small category and for $A \in \mathcal{C}$, $h^A := \mathrm{Hom}_{\mathcal{C}}(A,-)$ and $F : \mathcal{C} \to \underline{\mathrm{Set}}$ functors. Then there is a natural isomorphism

$$\mathrm{Nat}(h^A, F) \cong F(A)$$

given by $\alpha \longmapsto \alpha_A(\mathrm{id}_A)$.

Proof: exercise!

A representation is a natural isomorphism $h^A \to F$, and so by Yoneda, these are associated to some elements of $F(A)$. What are they?

Definition: A universal element of the functor $F : \mathcal{C} \to \underline{\mathrm{Set}}$ is a pair (A, u) where $A \in \mathcal{C}$ and $u \in F(A)$ such that for any other pair (B, v) where $B \in \mathcal{C}$, $v \in F(B)$. There exists a unique! morphism $\varphi : A \to B$ such that $v = F(\varphi)(u)$.

$$\begin{tikzcd} A \arrow[d, dashed, "\varphi"] & F(A) \arrow[d, dashed, "F(\varphi)"'] & u \arrow[d, mapsto] \\ B & F(B) & v \end{tikzcd}$$

Stupid example: Consider the functor $h^{\mathbb{Z}} : \underline{\mathrm{Ab}} \to \underline{\mathrm{Set}}$. Then $(\mathbb{Z}, \mathrm{id}_{\mathbb{Z}})$ is a universal element for this functor since suppose we have another pair (G, φ). So $\varphi \in \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, G)$

we have another pair (G, φ). So $\varphi \in \mathrm{Hom}_{Ab}(\mathbb{Z}, G)$ and we have $\varphi = \varphi \circ id_{\mathbb{Z}} = h^{\mathbb{Z}}(\varphi)(id_{\mathbb{Z}})$.

The point? representations correspond to universal elements.

<u>Lemma</u>: Let $F: \mathcal{C} \to \underline{Set}$ be a functor, then is a bijection

$$\left\{ \begin{matrix} \text{natural isom} \\ h^A \to F \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{universal elements} \\ \text{for } F \end{matrix} \right\}$$

and this bijection is given by the map $\alpha \mapsto (A, \alpha_A(id_A))$ as in the Yoneda Lemma.

<u>Proof</u>: Suppose $\alpha: h^A \to F$ is a natural transformation. Then for each $\varphi \in \mathrm{Hom}(A,B)$, by naturality we have

$$\begin{array}{ccc} \mathrm{Hom}(A,A) & \xrightarrow{\alpha_A} & F(A) \\ \downarrow h^A(\varphi) & & \downarrow F(\varphi) \\ \mathrm{Hom}(A,B) & \xrightarrow[\alpha_B]{} & F(B) \end{array} \quad (1)$$

If α is an isomorphism, then α_B is a bijection and so for any $v \in F(B)$, there exists a $\varphi' \in \mathrm{Hom}(A,B)$ such that $\alpha_B(\varphi') = v$. By (1), we have $v = \alpha_B(\varphi') = \alpha_B(\varphi' \circ id_A)$

$= F(\varphi')(\alpha_A(id_A))$. Hence $(A, \alpha_A(id_A))$ is a universal element.

Conversely, suppose $(A, \alpha_A(id_A))$ is a universal element. Then for any pair (B, v), there exists a unique $\varphi: A \to B$ such that $v = F(\varphi)(\alpha_A(id_A))$

$= \alpha_B(\varphi)$ by (1)

$$= F(f)(\alpha_A(1_A))$$

$$= \alpha_B(\varphi) \quad \text{by } (1)$$

Hence by uniqueness, α_B is a bijection and as B arbitrary, α is an isomorphism. ∎

<u>Useful Result:</u> Given a locally small category $\mathcal{C}$ and $A \in \mathcal{C}$, the functor $Hom_{\mathcal{C}}(A, -)$ preserves limits.

Question 1. Let G (not necessarily abelian) be a group and consider the functor $F : \boldsymbol{Ab} \to \boldsymbol{Set}$ given by $F(-) = Hom_{\boldsymbol{Grp}}(G, -)$. Is this functor representable?

Let $G^{ab} = G/[G,G]$. I claim F is representable by G^{ab}. For any abelian group H, and morphism $\varphi : G \to H$, this uniquely factors through the quotient $q : G \to G^{ab}$.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "q"'] & H \\ G^{ab} \arrow[ur, dashed, "\exists ! \bar{\varphi}"'] & \end{tikzcd}$$

Hence, let $\alpha_H : Hom(G^{ab}, H) \to Hom(G, H)$. This is a
$$h \mapsto q \circ h.$$
bijection and defines a natural isomorphism. Hence F is representable.

Note: $q : G \to G^{ab}$ is a universal element for F.

Question 2. Consider the functor $F : \boldsymbol{Grp} \to \boldsymbol{Set}$ given by $F(G) = \{g \in G \mid g^2 = e\}$. Is this functor representable? What about the functor $[\mathrm{tor}] : \boldsymbol{Grp} \to \boldsymbol{Set}$ given by $G[\mathrm{tor}] = \{g \in G \mid g^n = e \text{ for some } n\}$?

Let $\alpha_G : \mathrm{Hom}_{Grp}(\mathbb{Z}/2\mathbb{Z}, G) \longrightarrow F(G)$

$\varphi \longmapsto \varphi(1)$.

this is easily seen to be bijective and define a natural transformation. Hence representable. Note, the universal element is $(\mathbb{Z}/2\mathbb{Z}, 1)$

Now, suppose $[\mathrm{tor}]$ was representable. Then there exists a universal elemen (G, g) such that there exists a $\varphi : G \to \mathbb{Z}/n\mathbb{Z}$ such that $F(\varphi) : G[\mathrm{tor}] \to \mathbb{Z}/n\mathbb{Z}$ with $F(\varphi)(g) = 1$ ie $\varphi(g) = 1$. Hence n must divide the order of g. Since n arbitrary, no such universal element exists.

Question 3. Fix nonempty sets Y, Z and consider the contravariant functor $F : \boldsymbol{Set} \to \boldsymbol{Set}$ given by $F(X) = \mathrm{hom}(X, Y) \coprod \mathrm{hom}(X, Z)$. Is this functor representable?

Suppose it is representable. Then $F(-) \simeq \mathrm{Hom}(-, S)$ for some $S \in \underline{Set}$. Then

$$\mathrm{Hom}(\{*\}, S) \simeq \mathrm{Hom}(\{*\}, X) \sqcup \mathrm{Hom}(\{*\}, Y)$$

$$S \simeq X \sqcup Y.$$

But then $F(\{1,2\})$ are homs with image either completely in X or Y. While $\mathrm{Hom}(\{1,2\}, S)$ can be either.

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Question 4. Prove that representable functors preserve limits.

Suppose $\mathcal{C}$ is locally small and let $A \in \mathcal{C}$. Let J be index set and $F: J \to \mathcal{C}$ a diagram. Suppose we have a limiting cone $\nu: \varprojlim F \xrightarrow{\cdot} F$. Apply $\mathrm{Hom}(A, -)$ which gives us a cone.

$$\mathrm{Hom}_{\mathcal{C}}(A, \varprojlim F) \xrightarrow{\nu_i'} \mathrm{Hom}(A, F_i)$$

$$\|$$

$$X \xrightarrow{\tau_i} \mathrm{Hom}(A, F_i)$$

Suppose we have a cone $\tau_i: X \to \mathrm{Hom}(A, F_i)$. Then for each $x \in X$, $\tau_i x: A \to F_i$ and is a cone. Hence exists unique $h_x: A \to \varprojlim F$ such that $\tau_i x = \nu_i h_x$ for all $i \in J$

This gives a unique map $h: x \mapsto h_x$ such that $\tau_i = \nu_i' h$.
Hence $\varprojlim (A, F_i) \simeq \mathrm{Hom}(A, \varprojlim F)$

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