Week 8

Question 1. (Fall '11 Q2)

Let G be a nontrivial group and p a prime. If every subgroup $H \neq G$ has index divisible by p, prove the center of G is divisible by p. Note, I think the question wants to say for all nontrivial $H \neq G$.

If we take the question as stated and assume $|G:\{e\}|$ is divisible by p, then G is a p-group and this question is trivial. So assume it meant all nontrivial proper subgroups have index divisible by p.

Now, if G not a p-group (otherwise done), since $|G:Z(G)|$ is divisible by p, this implies $Z(G)$ is not trivial. Consider the action of $G \curvearrowright G$ given by conjugation. Since $Z(G)$ fixes G under this action, we get a well defined action $G/Z(G) \curvearrowright G$ and $G/Z(G)$ is a p-group, and so let $x_1, \dots, x_n$ be a complete set of representatives of orbits with nontrivial orbits. We get that

$$|G| = |Z(G)| + \sum \mathcal{O}(x_i)$$

and so by orbit-stabalizer, $|\mathcal{O}(x_i)|$ are divisible by p. $\Rightarrow$ $|Z(G)| \equiv 0 \mod p$

$p. \Rightarrow |Z(G)| \equiv 0 \mod p$

$\boxtimes$

Question 2. (Spring '07 G2)

How many subgroups of $\mathbb{Z}^n$ have index 5?

We will show a more general statement:
How many subgroups of $\mathbb{Z}^n$ have index p, for p prime.

Let H be a subgroup of index p. Then we have the quotient map $\mathbb{Z}^n \to \mathbb{Z}^n/H$ and $\mathbb{Z}^n/H \cong \mathbb{Z}/p\mathbb{Z}$. However, this isom is not unique. In particular, we get another isom by post composing with an element of $\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$ and every such isom is given by this.

What this gives us is the following: Let $\Omega = \mathrm{Hom}^{sur}(\mathbb{Z}^n, \mathbb{Z}/p\mathbb{Z})$ and $G = \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$.

Then $G \circlearrowright \Omega$ by post composition and we have a bijection between orbits of this action and subgroups of G of index p. Hence we use Burnside to count the orbits. Let $G = \langle \sigma \rangle$.

We have the following:

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We have the following:

$|\Omega^e| = |\Omega| = p^n - 1$. This follows as any surjective hom $\mathbb{Z}^n \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ is determined by where it sends any of the e_i vectors, and it's surjective as long as not all map to 0.

$|\Omega^\sigma| = \emptyset$ since the maps are surjective.

Hence $|\Omega/G| = \frac{1}{|G|}(p^n - 1) = \frac{p^n - 1}{p-1}$.

which for this question, gives us $\frac{5^n - 1}{4}$ subgroups

Question 3. (Fall '17 Q1)

Let G be a finite group, p a prime number and S a sylow p-subgroup of G. Let $N = \{g \in G \mid gSg^{-1} = S\}$. Let X and Y be subsets of $Z(S)$ such that there exists a $g \in G$ such that $gXg^{-1} = Y$. Show there exists an $n \in N$ such that $nxn^{-1} = gxg^{-1}$ for all $x \in X$.

We have an action $G \curvearrowright Z(S)$ given by conjugation. Then $g\,\mathrm{Stab}_G(X)g^{-1} = \mathrm{Stab}_G(Y)$ as

$$\begin{aligned} gfg^{-1} \cdot y &= gfg^{-1}ygf^{-1}g^{-1} \quad \text{for } f \in \mathrm{Stab}(X) \\ &= gg^{-1}ygg^{-1} \quad \text{as } g^{-1}yg \in X \\ &= y. \end{aligned}$$

Now, $S \leq \mathrm{Stab}_G(X)$ and $S \leq \mathrm{Stab}_G(Y)$ since both X, Y contained in the center. In particular, we have S and $g^{-1}Sg$ contained in $\mathrm{Stab}(X)$.

By Sylow 2, there exist a $h \in \mathrm{Stab}_G(X)$ such that

By Sylow 2, there exist a $h \in Stab_G(X)$ such that $S = h^{-1}g^{-1}Sgh \Rightarrow gh \in N$ and let $n = gh$.

Now, $nxn^{-1} = ghxh^{-1}g^{-1} = gxg^{-1}$ for all $x \in X$.

Question 4. (Spring '13, Q1)

Let G be a free abelian group of rank r. Show that G has only finitely many subgroups of a given finite index n.

Same idea as Q2. Since G abelian, any subgroup is normal, and so if H has index n, we have a hom $G \twoheadrightarrow G/H$

Let $\{K_i\}_{i \in I}$ be a complete set of representatives of isomorphic classes of abelian groups of order n.

We can then construct an injective map

$$A := \left\{ \begin{matrix} \text{subgroups of} \\ G \text{ of index } n \end{matrix} \right\} \hookrightarrow \left\{ \begin{matrix} \text{surjective hom} \\ G \to K_i \text{ for some } i \end{matrix} \right\} =: B.$$

Since there are only finite isomorphism classes of order n abelian groups (classification of f.g abelian groups) B is finite and so A is also finite.