Week 7

Burnside's Lemma

Given a G-set X, we have

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

<u>useful corollary</u>: Given a transitive G-set X with atleast 2 elements, there exists an element $g \in G$ such that $X^g = \phi$.

<u>proof:</u> Suppose we had $|X|^g \geq 1$ for all $g \in G$.

then
$$\frac{1}{|G|} \sum_{g \in G} |X^g| = \frac{1}{|G|}\left(|X^1| + \sum_{g \in G \setminus 1} |X^g|\right)$$
$$\geq \frac{1}{|G|}\left(2 + (|G|-1)\right)$$
$$> 1 = |X/G| \text{ as transitive.}$$

Hence we have a contradiction. ∎

Question 1. Let G be a finite group and $H < G$ a proper subgroup. Show that $\cup_{g \in G} gHg^{-1} \neq G$. Bonus question: What happens when G is infinite?

Let X be the set of all subgroups of G

[illegible]

Let X be the set of all subgroups of G conjugate to H. Then X is a transitive G-set and suppose $\bigcup_{g\in G} gHg^{-1} = G$. Then for all $g \in G$, there exists $h \in G$ such that $g \in hHh^{-1}$. Then we have $g \cdot hHh^{-1} = hHh^{-1}$ and so $|X^g| \geq 1$ for all $g \in G$. This contradicts the above corollary.

When G is infinite, this is not true. Take $T \subseteq GL_n(\mathbb{C})$ subgroup of upper triangular matrices. The Jordan-normal form implies that T's conjugates cover $GL_n(\mathbb{C})$.

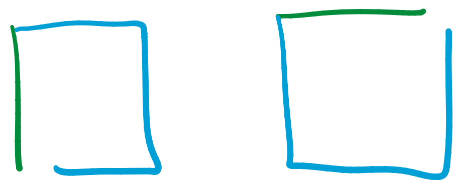
Question 2. Show that for a finite group G and proper subgroup H, there exists a conjugacy class of G that does not intersect H.

Let $g \in G \setminus \bigcup_{g\in G} gHg^{-1}$. This exists by the previous problem. Suppose there exists a $h \in G$ such that $hgh^{-1} \in H \Rightarrow g \in h^{-1}Hh$ but this can't be true. Hence $C_g \cap H = \emptyset$. $\blacksquare$

Use of Burnside in counting.

Suppose we want to count the number of ways

Suppose we want to count the number of ways we can colour the edges of a square with two colours upto rotation.

ie  are considered the same

Let X be the set of all configurations of colourings and $\sigma \in S_4$ acts on this set by 90° clockwise rotation. Hence we want to count the number of orbits! $|X/S_4|$, so we can use burnside for this, and instead count the number of configurations that are fixed by elements of S_4.

element of S_4	number fixed	squares left fixed.
e	16	all
σ	2	all edges same colour.
σ^2	4	opposite edges same colour
σ^3	2	all edges same colour.

Hence $|X/G| = \frac{1}{4}(16+2+4+2) = \frac{24}{4} = 6.$

colourings upto rotation.

Question 3. (Useful results for counting) Let G be a group and X a G-set. For $x, y \in G$, show that

(a) if x and y are conjugate then $|X^x| = |X^y|$.

(b) if x and y generate the same subgroup then $|X^x| = |X^y|$.

(a) If $a \in X^y$ and $x = hyh^{-1}$. Then $ha \in X^x$ since $x \cdot ha = hyh^{-1}ha = ha$. Hence we get a map $f_h : X^y \to X^x$ by $a \mapsto ha$ which has an inverse by h^{-1}. Therefore a bijection and so $|X^x| = |X^y|$

(b) if $a \in X^x$, then as $y = x^n$ for some n, so $a \in X^y$ $\Rightarrow X^x \subseteq X^y$ and by symmetry we get equality.

Question 4. Use Burnside's lemma to answer the following counting problem. Let n be an even number and and suppose we have n indistinguishable balls and put them into 3 indistinguishable jars. How many ways can we do this?

$X = \{(a,b,c) \in \mathbb{Z}^3_{\geq 0} \mid a+b+c = n\}$. and we have $S_3 \curvearrowright X$ given by $(12) \cdot (a,b,c) = (b,a,c)$

and $(123) \cdot (a,b,c) = (c,a,b)$.

We want to count X/S_3 and so by previous question we want to figure out fixed points for a representative for each conjugacy class a representative for each conjugacy class

element in conjugacy	# fixed	what gets fixed.
e	$\binom{n+2}{2}$	all of them
(12)	$n/2+1$	(a,a,b) for $a=0,\dots,\frac{n}{2}$
(123)	0	since n even, can't have one of form (a,a,a).

Hence Burnside gives us:

$$|X/S^3| = \frac{1}{6}\left(\binom{n+2}{2} + 3\left(\frac{n}{2}+1\right)\right).$$

Question 5.

(a) Let X be a finite G-set with $|G| = p^n$ for some prime p and p does not divide $|X|$. Show there exists an element $x \in X$ such that $gx = x$ for all $g \in G$.

(b) Let V be a d-dimensional vector space over $\mathbb{Z}_p$ and let $G \subset GL_d(\mathbb{Z}_p)$ be a group such that $|G| = p^n$. Show that there exists a nonzero vector $v \in V$ such that $g \cdot v = v$ for all $g \in G$.

Note, you don't need burnside to do these questions.

(a) Suppose no such element $x \in X$ exists. Then the size of every orbit is divisible by p (all orbits are nontrivial and orbit-stabalizer implies that there order divides p^n.)

Since $|X| = \sum |O_i|$ where O_i are the orbits, we have $|X| \equiv 0 \mod p$, but this is a contradiction.

(b). Take $X = V \setminus \{0\}$. Then $|X| = p^d - 1 \equiv -1 \mod p$.

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Hence apply the previous part.

Question 6. Prove the Frattini argument: Let G be a finite group and $H \triangleleft G$. Suppose P is a Sylow p-subgroup of H. Then $G = HN_G(P)$.

Since $H \triangleleft G$, we have $G \curvearrowright Syl_p(H)$ transitively. But by Sylow theorems, $H \curvearrowright Syl_p(H)$ also acts transitively. In particular, for all $g \in G$, there exists $h \in H$ such that $h^{-1} g P g^{-1} h = P$

$\Rightarrow h^{-1} g \in N_G(P) \Rightarrow g \in HN_G(P) \Rightarrow G = HN_G(P)$

$\blacksquare$