Week 6

Crash course on representation theory.

Let V be a (f.d) vector space over field k. A (linear) representation of finite group G is a homomorphism $\rho: G \to GL_k(V)$. This is the same thing as a G-action on V such that $g\cdot(v+w) = g\cdot v + g\cdot w$ and $g\cdot(\lambda v) = \lambda(g\cdot v)$. $\lambda \in k$.

Question 1. For each of the following representations $\rho : G \to GL(V)$, describe what the matrices $\rho(g)$ look like.

(a) V is a one dimensional vector space. Note, G is finite.

(b) Let X be a G-set. Then the action of G on X extends linearly to an action on $F(X)$. Take $V = F(X)$ and $\rho(g)$ is given by $\rho(g)(x) = g \cdot x$ on the basis $x \in X$.

(a) let $g \in G$, then exists n s.t $g^n = 1$. Hence

$$\rho(g)^n = \rho(g^n) = \rho(1) = I.$$

Since $V = k^\times$ in this case, it follows $\rho(g)$ are roots of unity.

(b) X forms a basis after some ordering, and $\rho(g)$ are then permutation matrices.

A bunch of definitions

(1) if $\rho: G \to GL(V)$ and $\rho': G \to GL(W)$ are two representations of G. Then $\rho \oplus \rho' : G \to GL(V \oplus W)$ is the representation given by

the representation given by

$$(\rho \oplus \rho')(g) = \rho(g) + \rho'(g)$$

② A subspace $W \leq V$ is G-invariant if $\forall g \in G$ $\rho(g)(W) \subseteq W$. Then ρ induces two representations. one on W given by the restriction, and one on the quotient V/W given by $\rho_{V/W}(g)(v+W) = \rho_V(g)(v) + W$.

③ A representation of G, $\rho: G \to GL(V)$ is irreducible if V has no nontrivial proper G-invariant subspace. otherwise reducible

④ A representation $\rho: G \to GL(V)$ is completely reducible if for all G-invariant subspces W, there exists another G-invariant subspace W' such that $W \oplus W' = V$.

⑤ Given two representation $\rho_1: G \to GL(V)$, $\rho_2: G \to GL(W)$, a G-linear map $\varphi: V \to W$ is a linear transformation such that $\varphi(\rho_1(g)v) = \rho_2(g)\varphi(v)$ for all $g \in G$, $v \in V$.

Question 2. We will prove Maschke's theorem: Let k be a field such that $|G|$ does not divide the order of k. Then any k-representation $\rho : G \to GL(V)$ is completely reducible. (If you don't know what characteristic is, take k to be $\mathbb{R}$ or $\mathbb{C}$)

(a) Let W be a G-invariant subspace of V and $\pi : V \to V$ any projection onto W. Define the following map $\pi' : V \to W$ by

$$\pi'(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)\pi(\rho(g^{-1})v).$$

Show that π' is also a projection onto W. A projection on W is a linear transformation such that $\pi^2 = \pi$ and $\mathrm{im}(\pi) = W$.

(b) Show that π' is a G-linear map.

(c) Show that there exists a G-invariant subspace W' such that $W \oplus W' = V$. That is, V is completely reducible.

(a) Let $v \in V$, then $\pi'(v) \in W$ since $\pi(\rho(g^{-1})v) \in W$.

Hence $\pi'(V) \subseteq W$. Let $v \in W$. Then

$$\pi'(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)\, \pi(\rho(g^{-1})v)$$

$$= \frac{1}{|G|} \sum_{g \in G} \rho(g)\rho(g^{-1})v$$

$$= \frac{1}{|G|} \sum_{g \in G} v = v.$$

Hence $\pi'|_W = \mathrm{id}_W$ and so we get $\pi'^2 = \pi$ and $\mathrm{im}\, \pi = W$.

(b) We want to show that $\pi'(\rho(g)v) = \rho(g)\pi'(v)$.

$$\pi'(\rho(g)v) = \frac{1}{|G|} \sum_{h \in G} \rho(h)\, \pi(\rho(h^{-1})\rho(g)v)$$

$$= \frac{1}{|G|} \sum_{h \in G} \rho(gh)\, \pi(\rho(h^{-1})v)$$

since $h \mapsto gh$ is a bijection.

$$= \frac{1}{|G|} \rho(g) \sum_{h \in G} \rho(h) \pi(\rho(h^{-1}) v)$$

$$= \rho(g) \pi'(v).$$

(c). let $W' = \ker \pi'$. Then we want to show that W' is G-invariant. Suppose $g \in G$, $v \in W'$

$\pi'(\rho(g)v) = \rho(g)\pi'(v) = 0$. Hence G-invariant.

we now show $W \oplus W' = V$.

let $v \in W \cap W'$, then $\pi'(v) = v$ since $v \in W$

$\pi'(v) = 0$ since $v \in W'$

$$\Rightarrow v = 0.$$

$$\Rightarrow W \cap W' = \{0\}.$$

Now, let $v \in V$. Then $v = \pi'(v) + v - \pi'(v)$.

$\pi'(v) \in W$ and $v - \pi'(v) \in W'$ since $\pi'(v - \pi'(v)) =$

$\pi'(v) - (\pi')^2(v) = \pi'(v) - \pi'(v) = 0$.

$\boxed{\square}$

Question 3. Let $G = \{1, x, x^2\}$ be the cyclic group of order 3 and define a complex representation $\rho : G \to GL(\mathbb{C}^3)$ by $\rho(x)(z_1, z_2, z_3) = (z_2, z_3, z_1)$. Find the irreducible G-invariant subspaces of $\mathbb{C}^3$. (There will only be three of them for reasons, what's special about linear operators over complex numbers?)

$\rho(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ with respect to the standard basis.

Moreover $\rho(x)^3 = I$ so has eigen values $1, \omega, \omega^2$ where ω primitive. cube root of unity

We find the corresponding eigenvalues

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ 1-eigenvector

$$\left(\begin{array}{ccc|c} -w & 1 & 0 & 0 \\ 0 & -w & 1 & 0 \\ 1 & 0 & -w & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & w^2 & -w & 0 \\ 0 & -w & 1 & 0 \\ 1 & 0 & -w & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -w & 0 \\ 0 & 1 & -w^2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So $\begin{pmatrix} w \\ w^2 \\ 1 \end{pmatrix}$ is w-eigenvector.

Similarly, $\begin{pmatrix} w^2 \\ w \\ 1 \end{pmatrix}$ is w^2-eigenvector.

Hence we have the decomposition $\mathbb{C}^3 = \mathbb{C}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} w \\ w^2 \\ 1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} w^2 \\ w \\ 1 \end{pmatrix}$

We can easily see that each of these eigenspaces are G-inv.

Question 4. We will prove Schur's lemma (Some version of it at least). Consider a irreducible complex representation $\rho : G \to V$. Let $\phi : V \to V$ be a G-linear map. Show there exists a scalar $\lambda \in \mathbb{C}$ such that $\phi(v) = \lambda v$ for all $v \in V$.

- $\mathrm{Im}\,\phi$ is a G-invariant subspace, if ϕ is a G-linear map.
  if $y = \phi(x)$, then for $g \in G$, $\rho(g)y = \rho(g)\phi(x)$
  $= \phi(\rho(g)x)$.

- $\mathrm{Ker}\,\phi$ is a G-invariant subspace. If $\phi(x) = 0$, then
  $\phi(\rho(g)x) = \rho(g)\phi(x) = 0$.

- Consider $\psi = \phi - \lambda I$. For some $\lambda \in \mathbb{C}$, ψ has nontrivial kernel. Since V is irreducible, $\psi = 0$ on V. Hence $\phi = \lambda I$.

Question 5. Let G be an abelian group. Prove all irreducible complex representations of G are one-dimensional.

Suppose $\rho: G \to GL(V)$ is an irreducible complex representation. For $g \in G$, let $\phi_g : V \to V$ be given by $\phi_g(v) = \rho(g)v$. Since G Abelian, this determines a G-linear map.

$$\phi_g(\rho(h)v) = \rho(g)\rho(h)v = \rho(h)\phi_g(v).$$

Hence Schur's lemma tells us that $\phi_g(v) = \lambda_g v$ for some scalar $\lambda_g \in \mathbb{C}$. Since this holds for all G, we conclude V is 1-dimensional.