Week 4 Notes

Theorem 1. *The following a equivalent*

(a) Group G is nilpotent

(b) If H is a proper subgroup of G, then H is a proper subgroup of $N_G(H)$. That is, normalizers grow.

(c) Every Sylow subgroup is normal.

(d) G is the direct product of Sylow subgroups.

(e) If d divides $|G|$, then G contains a normal subgroup of order d.

Proof of Theorem 1 (For completeness)

We will prove the following implications

$$\begin{tikzcd}
(a) \arrow[d, Rightarrow] & (d) \arrow[l, Rightarrow] \arrow[dr, Rightarrow] & \\
(b) \arrow[r, Rightarrow] & (c) \arrow[u, Rightarrow] & (e) \arrow[l, Rightarrow]
\end{tikzcd}$$

(a) $\Rightarrow$ (b)

If G is nilpotent and $H<G$ a proper subgroup. Then given a central series $\{e\}=G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$, there exists i s.t $G_i < H$ and $G_{i+1} \not< H$. Let $x \in G_{i+1} \setminus H$ and $h \in H$. Then $[h,x] \in G_i \leq H$ ie $hxh^{-1}x^{-1} \in H$ so inverting, $xhx^{-1}h^{-1} \in H \Rightarrow xhx^{-1} \in H \Rightarrow x \in N_G(H)$. Hence $H < N_G(H)$ proper.

(b) $\Rightarrow$ (c): Let P be a Sylow p-subgroup. Let $g \in N_G(N_G(P))$ then $gN_G(P)g^{-1} = N_G(P) \Rightarrow gPg^{-1} \leq N_G(P)$

$$\Rightarrow gPg^{-1} = P \text{ since Sylow theorem}$$

implies only one sylow subgroup in it's normaliser. Hence we have $N_G(P) = N_G(N_G(P))$ and so if (b) holds, $N_G(P) = G$.

(c) ⇒ (d): Sylow theorem implies a unique Sylow p-subgroup for each prime divisor. Lagrange implies trivial intersection. So if $|G| = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ and $\underline{P}_i$ the unique Sylow p_i-subgroup. We have $|G| = |\underline{P}_1||\underline{P}_2| \cdots |\underline{P}_n|$. Hence G is the internal direct product $G = \underline{P}_1 \times \cdots \times \underline{P}_n$.

(d) ⇒ (a):

p-groups are nilpotent. This follows as p-groups have nontrivial centers, so the upper central series terminates. Also, direct sums of nilpotent groups are nilpotent.

(d) ⇒ (e): Note, p-groups contain normal subgroups of all orders. This follows as they have nontrivial centers. (so consider quotients and pull back).

(e) ⇒ (c): clear.

Question 1. (Why the heck are they called nilpotent groups anyway?) Consider the map $\mathrm{adj}_g : G \to G$ given by $\mathrm{adj}_g(x) = [g, x]$. This is called the adjoint map. Show that if G is nilpotent, then there exists some n such that $\mathrm{adj}_g^n(x) = e$ for all $x, g \in G$.

We have a normal series $\{e\} = G_n \trianglelefteq G_{n-1} \trianglelefteq \cdots \trianglelefteq G_0 = G$. with $[G, G_i] \leq G_{i+1}$

Hence, for any $g, x \in G$, $\mathrm{adj}_g^i(x) \in G_i$. This follows from induction We have $\mathrm{adj}_g(x) = [g, x] \in G_1$ and $\mathrm{adj}_g^{i+1}(x) = [g, \mathrm{adj}_g^i(x)] \in G_{i+1}$

Hence, for any $g, x \in G$, $\mathrm{adj}_g(x) \in G_1$. This follows from induction

We have $\mathrm{adj}_g(x) = [g, x] \in G_1$ and $\mathrm{adj}_g^{i+1}(x) = [g, \mathrm{adj}_g^i(x)] \in G_{i+1}$.

Hence, it follows that $\mathrm{adj}_g^n(x) = e$ for all $x, g \in G$.

Question 2. Consider the quarternion group

$$Q_8 = \langle -1, i, j, k \mid i^2 = j^2 = k^2 = ijk = -1, (-1)^2 = 1 \rangle.$$

Is this nilpotent?

From H/W, we showed all subgroups were normal. So by Thm 1(b). Q_8 is nilpotent.

Question 3. Prove that G is nilpotent if and only if $G/Z(G)$ is nilpotent.

($\Rightarrow$) Suppose $H/Z(G) \leq G/Z(G)$ proper subgroup where $Z(G) \subseteq H$. Then let $N/Z(G)$ be the normaliser of $H/Z(G)$ where N is a subgroup of G that contains $Z(G)$. Suppose $n \in N_G(H)$. Then $nZ(G)H(nZ(G))^{-1} = nHn^{-1} = H$. So we have $N_G(H) \subseteq N$. Since G nilpotent, H is then proper in N and $H/Z(G)$ proper in $N/Z(G)$.

Hence $G/Z(G)$ is nilpotent.

($\Leftarrow$) The same setup as last part. This time we have $H/Z(G)$ proper in $N/Z(G)$ and so by bijective correspondence, H proper in N. WTS: $N \subseteq N_G(H)$.

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let $n \in N$. Then $nHn^{-1} = (nZ(G))H(nZ(G))^{-1} = H$.
and so $N \subseteq N_G(H)$ and we are done.

<u>Alternate proof using upper central series:</u>

if G nilpotent, the upper central series terminates

$$\{e\} \trianglelefteq Z(G) \trianglelefteq Z_2 \trianglelefteq Z_3 \trianglelefteq \cdots \trianglelefteq Z_n = G$$

Then we have

$$\{e\} \trianglelefteq Z_2/Z(G) \trianglelefteq Z_3/Z(G) \trianglelefteq \cdots \trianglelefteq G/Z(G)$$

which is the upper central series of $G/Z(G)$ (3rd isom)

Hence $G/Z(G)$ is nilpotent.

conversely, if $G/Z(G)$ nilpotent, then upper central series terminates:

$$\{e\} = Z_0 \trianglelefteq Z_1 \trianglelefteq \cdots \trianglelefteq Z_n = G/Z(G).$$

Let $q: G \to G/Z(G)$ be the quotient map. the normal series

$$\{e\} = Z(G) \trianglelefteq q^{-1}Z_0 \trianglelefteq q^{-1}Z_1 \trianglelefteq \cdots \trianglelefteq q^{-1}(Z_n) = G$$

is the upper central series. Hence G nilpotent.

Question 4. Prove that the dihedral group

$$D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$$

is nilpotent if and only if $n = 2^k$ for $k \in \mathbb{N}$.

Every element of D_n can be written uniquely in the form $a^k b^\ell$ when $k \in [n-1] (= \{0, 1, \ldots, n-1\})$, $\ell \in \{0,1\}$.

let $x \in Z(D_n)$, so $x = a^k b^\ell$ for some k, ℓ.

Suppose $\ell = 1$. Then $xa^s = a^s x \iff a^k b a^s = a^{k+s} b$

$$a^{k-s} b = a^{k+s} b$$

$$a^{-s} = a^s$$

and so we must have $a^{-s} = a^s$ for all $s \in [n-1]$ which is impossible. Hence $\ell = 0$. Now, consider general $y = a^s b^t$

then $xy = yx \iff a^k a^s b^t = a^s b^t a^k$

$$\iff a^{k+s} b^t = a^{s-k} b^t$$

$$\iff a^{2k} = e$$

so we must have that a^k s.t $(a^k)^2 = e$. Hence we get the center of D_n to be:

$$Z(D_n) = \begin{cases} \{e\} & \text{if } n \text{ odd} \\ \{e, a^{n/2}\} & \text{if } n \text{ even.} \end{cases}$$

Now, I claim that D_{2^n} is nilpotent. We show this via induction. D_1 is nilpotent as this is $\mathbb{Z}_2$. Now,

$D_{2^{n+1}}$ is nilpotent since $D_{2^{n+1}} / Z(D_{2^{n+1}}) \cong D_{2^n}$ which is nilpotent and the last question implies $D_{2^{n+1}}$ is nilpotent.

Conversely, suppose $n = 2^k m$ with m odd and $k \geq 0$.

Then the last question implies D_n nilpotent iff D_m is

Then the last question implies D_n nilpotent iff D_m is nilpotent. Suppose D_m was nilpotent. Then there exists a central series:

$$\{e\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_s = D_m.$$

with $H_1 \leq Z(D_m)$. But as m odd, $Z(D_m)$ is trivial and so $H_1 = H_0$. By induction, we get that all H_i are trivial and so no central series can exist. Hence D_m is not nilpotent.

Question 5. The derived series of a group G is the normal series

$$G = G^{(0)} \triangleright G^{(1)} \triangleright \cdots$$

where $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. Show that a group is solvable if and only if the derived series eventually terminates at $\{e\}$.

If the derived series terminates, then as $G^{(i-1)}/G^{(i)}$ abelian, this implies that G is solvable. Conversely, if G solvable, then we have a subnormal series:

$$\{e\} = H_n \trianglelefteq H_{n-1} \trianglelefteq \cdots \cdots \trianglelefteq H_0 = G.$$

such that H_{i-1}/H_i are abelian. In particular, this implies that as $H_0/H_1 = G/H_1$ abelian, $G^{(1)} = [G, G] \leq H_1$.

Suppose in general we have $G^{(i)} \leq H_i$. Then as H_i/H_{i+1} abelian, $[H_i, H_i] \leq H_{i+1} \Rightarrow G^{(i+1)} = [G^{(i)}, G^{(i)}] \leq H_{i+1}$.

Hence by induction, $G^{(i)} \leq H_i$ for all i and so $G^{(n)} \leq H_n = \{e\}$

$\blacksquare$

Question 6. Show that nilpotent groups are solvable. Can you think of an example of a solvable group that is not nilpotent?

If G is nilpotent, then we have central series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

where $[G, G_i] \leq G_{i+1}$. We have that $G^{(1)} = G_1 = [G,G]$ and suppose $G^{(i)} \leq G_i$. Then $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ $\leq [G, G_i] \leq G_{i+1}$. Hence by induction $G^{(i)} \leq G_i$ for all i. Therefore $G^{(n)} = \{e\}$ and so G is solvable.

Consider the group S_3. What is $S_3^{(1)} = [S_3, S_3]$? One characterisation of $S_3^{(1)}$ is the smallest normal subgroup such that the quotient is abelian. S_3 has only the trivial normal subgroup and $\langle (123) \rangle$, and S_3 itself. However, $|S_3 / \langle (123) \rangle| = 2$ and so is abelian. Hence, $[S_3, S_3] = \langle (123) \rangle$ and as this is abelian, we get S_3 is solvable.

Now, $Z(S_3) = \{e\}$ and so by similar argument as in the dihedral case, it is not nilpotent.

Question 7. Show that all groups of order < 60 are solvable.

If you already know that A_5 is the smallest simple nonabelian group (which has order 60), then any composition series for a group G s.t. $|G| < 60$, A_5 cannot be a factor and so all its composition factors must be any composition series for a group G s.t $|G| < 60$, A_5 cannot be a factor and so all it's composition factors must be abelian. Hence G solvable