Week 3 Notes.

Reminder:

Sylow theorems (Application of group actions)

__Sylow 1__: Given a group G and a prime divisor p of $|G|$. There exists a sylow p-subgroup of G.

__Sylow 2__: For each prime p, the sylow p-subgroups of G are conjugate to each other.

__Sylow 3__: If $|G| = p^n m$ where p prime, $n>0$ and $p \nmid m$. Let n_p be the number of sylow p-subgroups of G.
Then
1) $n_p \mid m$
2) $n_p \equiv 1 \pmod{p}$
3) $n_p = |G : N_G(P)|$ when $P \in Syl_p(G)$

It's useful to realize that all of these can be proven via group actions:

~~S1: via a p-subgroup H on set G/H via left mult.~~
~~S2: p-Sylow subgroup P on set G/P via left mult.~~
~~S3:~~

Table:

Theorem	Group	Set	Action
S1	H p-subgroup	G/H	left mult.
S2	$P \in Syl_p(G)$	G/P	left mult.
S3(1)	G	$Syl_p(G)$	conjugation
S3(2)	$P \in Syl_p(G)$	$Syl_p(G)$	conjugation
S3(3)	G	$Syl_p(G)$	conjugation.

Question 1

We have the factorization $18 = 2 \cdot 3^2$.

From Sylow 3 we have that $n_3 \equiv 1 \mod 3$ and $n_3 \mid 2$. Hence $n_3 = 1$ and so there exists a normal subgroup of order 9.

Question 2

~~We observe that we may assume~~
Note $P \triangleleft N_G(P)$. Hence we have $N_G(P)/P$ a well defined quotient group. Now $HP/P \leq N_G(P)/P$ and by the second isomorphism theorem we have

$$|HP/P| = |H/H \cap P|.$$

Hence it follows by Lagrange's theorem that

$$|H/H\cap P| \,\Big|\, |N_G(P)/P| \text{ and } |H/H\cap P| \,\Big|\, |H|.$$

As $p \nmid |N_G(P)/P|$ and H p-group. It follows that

$H/H\cap P$ is trivial and so $H \subseteq P$.

Alternatively, if you are willing to accept that every p-subgroup ~~H~~ is contained in a Sylow p-subgroup Q.
~~Then Sylow 2 implies there exists $g \in G$~~

(such that $H \leq Q$)

Then there exists a sylow p-subgroup Q of $N_G(P)$ that must have the same order as P. by Sylow 2, there exists $g \in N_G(P)$ such that $gPg^{-1} = Q$, but as normalizer $P = Q$.

<u>Lemma</u>: Let H be a p-subgroup of finite group G such that H isn't a Sylow p-subgroup. Then there exists $P \in Syl_p(G)$ such that $H \leq P$.

<u>Proof</u>:

First observe that since $hgH = gH \Leftrightarrow g^{-1}hg \in H$, it follows that $N_G(H) = \bigcup_{gH \in Fix(H) = (G/H)^H} gH$. That is, the normalizer is the union of all cosets fixed by ~~H~~ H by left multiplication. Hence we have that since ~~$|G/H|$~~ $|N_G(H)/H| \equiv |(G/H)^H| \equiv |G/H| \mod p$.

Since ~~H~~ not sylow, $|G/H|$ is divisible by p and so $|N_G(H)/H| \equiv 0 \mod p$. Hence, by cauchy's theorem and lattice theorem, there exists p-subgroup $H' \leq G$ such that $H \lneq H'$.

$\blacksquare$

Question 3

Suppose $Q \in Syl_p(G)$ is such that $pQp^{-1} = Q$ for all $p \in P$. Then $P \subseteq N_G(Q)$ and by the previous Q we have $P \subseteq Q$. Hence $P = Q$.

Question 4

The action gives us a group hom $G \to Sym(Syl_p(G))$

$\cong S_{n_p}$. Since G is simple, this mapping must

be injective and so by Lagrange's thm. $|G| \mid n_p!$

Question 5

Consider the left action of G on the left cosets of a proper subgroup H. Let $n = |G:H|$, Since G is simple, we have an injective hom $G \hookrightarrow S_n$. We want to show $10 \leq n$.

Now, since G has an element of order 21, S_n must contain a ~~cycle~~ element of order 21. Since an order of a permutation is equal to the lowest common multiple of cycle lengths in cycle decomp., and $21 = 3 \cdot 7$ we see that n must be atleast $3+7 = 10$.

Question 6

We first check that $N_G(P)$ acts on X^P.

Let $y \in X^P$. ie $gy = y$ for all $g \in P$. We want to show that $g'y \in X^P$ for $g' \in N_G(P)$.

Now, $g'^{-1}gg'y = y$ since $g'^{-1}gg' \in P$ for $g' \in N_G(P)$, $g \in P$.

Hence $gg'y = g'y$ and so $g'y \in X^P$.

Now, suppose $y \in X^P$. We already have $x \in X^P$ since $P \leq G_x$. It is sufficient to show that there exists $g \in N_G(P)$ such that $gx = y$.

Since $G \curvearrowright X$ is transitive, there exists $g \in G$ such that $gx = y$. Then we have $G_y = gG_xg^{-1}$

and so $g^{-1}Pg \leq G_x$ since $P \leq G_y$. ($y \in X^P$).

Hence by Sylow 2, there exists $h \in G_x$ such that $h^{-1}g^{-1}Pgh = P \Rightarrow gh \in N_G(P)$ and we have

$ghx = gx = y$. ∎