Week 2 Notes.

Some definitions:

- A subnormal series of a group G is a sequence of subgroups, each a normal subgroup of the next one. ie $1 = A_0 \trianglelefteq A_1 \trianglelefteq A_2 \trianglelefteq \cdots \trianglelefteq A_n = G$. (assume without repetition)

- The quotients A_{i+1}/A_i are called the factor groups of the series.

- We say two subnormal series of G are equivalent if there is a bijection between the sets of their factor groups such that the corresponding factor groups are isomorphic. That is, suppose we have two subnormal series:
$$1 = A_0 \trianglelefteq A_1 \trianglelefteq \cdots \trianglelefteq A_n = G$$
$$1 = B_0 \trianglelefteq B_1 \trianglelefteq \cdots \trianglelefteq B_m = G.$$
They are equivalent if $n=m$ and there exists a permutation $\pi \in S_n$ such that
$$A_{\pi(i)}/A_{\pi(i)-1} \cong B_i/B_{i-1} \quad \text{for all } i.$$

- A subnormal series where every factor group is simple is called a composition series and the factors composition factors

Question 1

Since G finite, there must exist a proper maximal subgroup $G_1 \trianglelefteq G$. Similarly, maximal proper subgroup $G_2 \trianglelefteq G_1$ and we repeat to get the subnormal series $G \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots$

Since G finite, this must terminate at 1. Hence we get a composition series

$$1 \trianglelefteq G_n \trianglelefteq G_{n-1} \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G$$

Note, by construction G_{i-1}/G_i are simple. This follows from bijection between ~~quot~~ subgroups of quotients and subgroups that contain that subgroup.

More generally, suppose we have a subnormal series:

$$1 = A_0 \trianglelefteq A_1 \trianglelefteq \cdots \trianglelefteq A_n = A.$$

Then A_i/A_{i-1} has a composition series and by the bijection, there exists a subnormal series

$$1 \trianglelefteq A_{i-1} = A_1^i \trianglelefteq A_2^i \trianglelefteq \cdots \trianglelefteq A_{n_i}^i = A_i \trianglelefteq G.$$

where A_j^i/A_{j-1}^i are all simple.

Hence we get a composition series of G.

$$1 \trianglelefteq A_1^0 \trianglelefteq \cdots \trianglelefteq A_{n_0}^0 = A_1^1 \trianglelefteq A_2^1 \trianglelefteq \cdots \trianglelefteq G.$$

Question 2

The second isomorphism theorem gives us that
$$AB/B \cong A/A \cap B.$$

However, AB/B is a normal subgroup of G/B which is simple. Hence $AB/B = G/B$, and so
$$G/B \cong A/A \cap B.$$

By symmetry, $G/A \cong B/A \cap B$.

Question 3: Let G be a finite group. If G has composition series of length 1, then it is simple and JH trivial in this case. Now, suppose G has shortest composition series given by:
$$1 = A_0 \trianglelefteq A_2 \trianglelefteq \cdots \trianglelefteq A_n = G$$
Suppose it has another composition series given by:
$$1 = B_0 \trianglelefteq B_1 \trianglelefteq \cdots \trianglelefteq B_m = G$$
If $A_{n-1} = B_{n-1}$ then by induction on series of A_{n-1}, the theorem follows. Hence suppose $A_{n-1} \neq B_{m-1}$.
$$C = B_{m-1} \cap A_{n-1}$$
Let $C = B_{m-1} \cap A_{n-1}$. This is normal in G and by previous question we have
$$A_{n-1}/C \cong G/B_{m-1} \quad \text{and} \quad B_{m-1}/C \cong G/A_{n-1}.$$

which are simple.

Now, since $A_{n-1} \trianglerighteq C$ we can extend this to a composition series of A_{n-1} which by induction must have length $n-1$. We then have the two composition series

$$1 = A_0 \trianglelefteq A_1 \trianglelefteq A_2 \trianglelefteq \cdots \trianglelefteq A_{n-2} \trianglelefteq A_{n-1}$$
$$1 = C_0 \trianglelefteq C_1 \trianglelefteq \cdots \quad \trianglelefteq C_{n-2} := C \trianglelefteq A_{n-1}.$$

which are equivalent by induction.

Now, $C_{n-2} \trianglelefteq B_{n-1}$ with simple quotient. Hence we have the two composition series for B_{m-1}:

$$1 = C_0 \trianglelefteq C_1 \trianglelefteq \cdots \trianglelefteq C_{n-2} \trianglelefteq B_{m-1}$$
$$1 = B_0 \trianglelefteq B_1 \trianglelefteq \cdots \trianglelefteq B_{m-2} \trianglelefteq B_{m-1}.$$

Hence, B_{m-1} has composition series of length $n-1$ which must be the minimum by minimality of n. So by induction, $m-1 = n-1$ and both composition series for B_{m-1} are equivalent.

Hence, it is equivalent to show that the series

$$1 = C_0 \trianglelefteq C_1 \trianglelefteq \cdots \trianglelefteq C_{n-2} \trianglelefteq B_{n-1} \trianglelefteq G$$
$$1 = C_0 \trianglelefteq C_2 \trianglelefteq \cdots \trianglelefteq C_{n-2} \trianglelefteq A_{n-1} \trianglelefteq G$$

are equivalent. But this follows by the previous question.

Definition: A group G is solvable if it has a subnormal series with abelian factor groups.

Question 4: Let $n \in \mathbb{N}$ and suppose
$n = p_1 p_2 \cdots p_m$ and $n = p_1' p_2' \cdots p_{m'}'$ where p_i, p_i' prime.

We have two different composition series for $\mathbb{Z}/n\mathbb{Z}$.

① $1 \trianglelefteq p_2 \cdots p_m\mathbb{Z}/n\mathbb{Z} \trianglelefteq p_3 \cdots p_m\mathbb{Z}/n\mathbb{Z} \trianglelefteq \cdots \trianglelefteq p_m\mathbb{Z}/n\mathbb{Z} \trianglelefteq \mathbb{Z}/n\mathbb{Z}$

② $1 \trianglelefteq p_2' \cdots p_{m'}'\mathbb{Z}/n\mathbb{Z} \trianglelefteq \cdots \trianglelefteq p_{m'}'\mathbb{Z}/n\mathbb{Z} \trianglelefteq \mathbb{Z}/n\mathbb{Z}$.

Note, the factor groups of ① are: (via 3rd isom)

$$\frac{p_i \cdots p_m\mathbb{Z}/n\mathbb{Z}}{p_{i-1} \cdots p_m\mathbb{Z}/n\mathbb{Z}} \cong p_i \cdots p_m\mathbb{Z}/p_{i-1} \cdots p_m\mathbb{Z} \quad \text{(3rd isom)}$$

$$\cong \mathbb{Z}/p_{i-1}\mathbb{Z} \quad \left(\text{take map } \mathbb{Z} \to p_i \cdots p_m\mathbb{Z}/p_{i-1} \cdots p_m\mathbb{Z} \text{ and use 1st isom}\right)$$

Similarly for factor groups of ②. Hence Jordan-Hölder implies there is a bijection
$\varphi : [m] \to [m']$ such that $p'_{\varphi(i)} = p_i \ \forall i$.
Hence the fundamental theorem of arithmetic follows.

Question 5:

composition factors of abelian groups are simple and Abelian. Hence they must ~~by~~ be cyclic primes.

If G is solvable

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

where G_i / G_{i-1} abelian. Any refinement of this to a composition series is such that any composition factor must also be a composition factor to one of the abelian groups G_i / G_{i-1}. Hence must be cyclic of prime order also.

~~Question 6: we have $1 \ntrianglelefteq H \trianglelefteq G$ which we can refine to some composition series~~

~~$1 \trianglelefteq H_0 \trianglelefteq$~~

~~$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = H \trianglelefteq G_0 \trianglelefteq \cdots \trianglelefteq G_m = G.$~~

~~Then~~